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On The Strong Metric Dimension of Lollipop Graph and Generalized Web Graph

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Abstract. ***Let*** G ***be a connected graph with vertex set*** $V(G)$ ***and edge set*** $E(G)$***. The interval*** $I[u; v]$ ***between*** u ***and*** v ***to be the collection of all vertices that belong to some shortest*** u v ***path. A vertex*** s ***strongly resolves two vertices*** u ***and*** v ***if*** u ***belongs to a shortest*** v s ***path, denoted by*** u 2 $I[v; s]$ ***or*** v ***belongs to a shortest*** u s ***path, denoted by*** v 2 $I[u; s]$***. A vertex set*** S ***of*** G ***is a strong resolving set of*** G ***if every two distinct vertices of*** G ***are strongly resolved by some vertex of*** S***. The strong metric basis of*** G ***is a strong resolving set with minimal cardinality. The strong metric dimension*** $sdim(G)$ ***of a graph*** G ***is defined as the cardinality of strong metric basis. In this paper we determine the strong metric dimension of a lollipop*** $L_{m;n}$ ***graph and a generalized web*** $WB(G; m; n)$ ***graph. Lollipop graph*** $L_{m;n}$ ***is the graph obtained by joining a complete graph*** K_m (m ***3) to a path graph*** P_n (n ***1) with a bridge. We obtain the strong metric dimension of a lollipop graph*** $L_{m;n}$ ***is*** m ***1. Generalized web graph*** $WB(G; m; n)$ ***is the graph obtained from the generalized pyramid graph*** $P(G; m)$ ***by taking*** p ***copies of*** P_n (n ***2) and merging an end vertex of a different copy of*** P_n ***with each vertex of the furthermost copy of*** G ***from the apex. We obtain the strong metric dimension of generalized web graph with*** G

C ***and without center vertex is*** $m = {}_m$

1. Introduction

Let G be a simple connected undirected graph $G = (V; E)$, where V is a set of vertices, and E is a set of edges. The distance between vertices u and v, i.e. the length of a shortest u-v path is denoted by $d(u; v)$. A vertex set $B=\{x_1; x_2; : : : ; x_k\}$ of G is a resolving set of G if every two distinct vertices of G are resolved by some vertex of B. The metric basis of G is a resolving set with minimal cardinality. The metric dimension of G, denoted by $dim(G)$ is the cardinality of its metric basis.

The concept of strong metric dimension was introduced by Seb¨o and Tannier [6] in 2004. Kratica et al. [2] defined for two vertices u and v in a connected graph G, if u belongs to a shortest $v - s$ path, denoted by $u \in I[v; s]$ or v belongs to a shortest $u - s$ path, denoted by $v \in I[u; s]$ then a vertex s strongly resolves two vertices u and v. A set of vertices $S \subseteq V(G)$ is strong resolving set for G if every two distinct vertices $u,v \in V(G)$ are strongly resolved by some vertex of S. The strong metric dimension of G denoted by $sdim(G)$ is the minimum cardinality over all strong resolving sets of G.

Many researchers have investigated the strong metric dimension to some graph classes. In 2004 Seb¨o and Tannier [6] observed that the strong metric dimension of complete graph K_n is n-1, cycle graph C_n is $\lceil \frac{n}{2} \rceil$, and tree is $L(T)$ - 1, where $L(T)$ denotes the number of leaves

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of tree. In 2013 Yi [8] determined that the strong metric dimension of G is 1 if and only if $G = P_n$. Kusmayadi et al. [3] determined the strong metric dimension of some related wheel graph such as sunflower graph, t-fold wheel graph, helm graph, and friendship graph. In this paper, we determine the strong metric dimension of a lollipop $L_{m;n}$ graph and a generalized web $WB(G; m; n)$ graph.

2. Main Result

2.1. Strong Metric Dimension

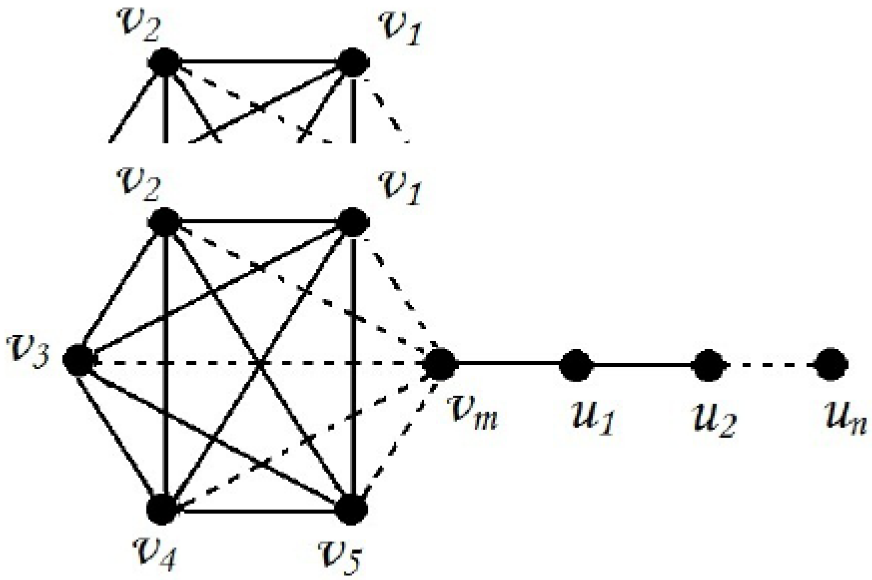
Let G be a connected graph with vertex set $V(G)$, edge set $E(G)$, and $S = \{s_1; s_2; : : : ; s_k\} \subseteq V(G)$. Oelermann and Peters-Fransen [5] defined the interval $I[u; v]$ between u and v to be the collection of all vertices that belong to some shortest $u - v$ path. A vertex $s \in S$ strongly resolves two vertices u and v if $u \in I[v; s]$ or $v \in I[u; s]$. A vertex set S of G is a strong resolving set of G if every two distinct vertices of G are strongly resolved by some vertex of S. The strong metric basis of G is a strong resolving set with minimal cardinality. The strong metric dimension of a graph G is defined as the cardinality of strong metric basis denoted by $sdim(G)$. We often make use of the following lemma and properties about strong metric dimension given by Kratica et al. [1].

Lemma 2.1 *Let $u, v \in V(G)$, $u \neq v$, and*

(i) $d(w,v) \leq d(u,v)$ for each w such that $\{w,u\} \in E(G)$, and

(ii) $d(u,w) \leq d(u,v)$ for each w such that $\{v,w\} \in E(G)$.

Then, there does not exist vertex $a \in V(G)$, $a \neq u,v$ that strongly resolves vertices u and v.



ery two vertices $u,v \in V(G)$

etween two vertices in G.

ry two vertices $u; v \in V(G)$

graph obtained by joining a

$L_{m;n}$ can be depicted as in

Figure 1. Lollipop graph $L_{m;n}$

Lemma 2.2 *For every integer $m \geq 3$ and $n \geq 1$, if S is a strong resolving set of lollipop graph $L_{m;n}$ then $|S| \geq m - 1$.*

Proof. Consider a pair of vertex $(v_i; u_n)$ with $i \in [1,m\text{-}1]$ satisfying condition $d(v_i; u_n) = n+1 = diam(L_{m;n})$. According to Property 2.2, we obtain $v_i \in S$ or $u_n \in S$, therefore S has at least $m - 1$ vertices. Hence $|S| \geq m - 1$.

Lemma 2.3 *For every integer $m \geq 3$ and $n \geq 1$, a set $S = \{v_1; v_2; : : : ; v_{m\ 1}\}$ is a strong resolving set of lollipop graph $L_{m;n}$.*

Proof. For every integer $i \in [1, m\text{-}1]$ satisfying condition $d(u_n; v_i) = n+1 = diam(L_{m;n})$, we obtain the shortest $u_n - v_i$ path: $u_n; u_{n\ 1}; u_{n\ 2}; : : : ; u_1; v_m; v_i$. So that v_i strongly resolves a pair of vertices (v_m, u_n). Thus $v_m \in I[u_n; v_i]$.

For a pair of vertices $(u_k; u_l)$ with $k \neq l$, $k; l \in [1,n\text{-}1]$, without loss of generality $k < l$ we obtain for $i \in [1,m\text{-}1]$, the shortest $u_l - v_i$ path: $u_l; u_{l\ 1}; : : : ; u_k; u_{k\ 1}; : : : ; u_1; v_m; v_i$. So that v_i strongly resolves a pair of vertices $(u_k; u_l)$. Thus $u_k \in I[u_l; v_i]$.

Therefore $S = \{v_1; v_2; : : : ; v_{m\ 1}\}$ is a strong resolving set of lollipop graph $L_{m;n}$.

Theorem 2.1 *Let $L_{m;n}$ be the lollipop graph with $m \geq 3$ and $n \geq 1$. Then $sdim(L_{m;n}) = m -$* 1.

Proof. By using Lemma 2.3 a set $S = \{v_1; v_2; : : : ; v_{m\ 1}\}$ is a strong resolving set of lollipop graph $L_{m;n}$ with $m \geq 3$ and $n \geq 1$. According to Lemma 2.2, $|S| \geq m$ - 1 so that S is a strong metric basis of lollipop graph $L_{m;n}$. Hence $sdim(L_{m;n}) = m$ - 1.

2.3. The Strong Metric Dimension of Generalized Web Graph

Miller et al. [4] defined the generalized web graph $WB(G; m; n)$ for $m \geq 3$, $n \geq 2$ is the graph obtained from the generalized pyramid $P(G; m)$ by taking p copies of P_n $(n \geq 2)$ and merging an end vertex of a different copy of P_n with each vertex of the furthermost copy of G from the apex.

In this paper we obtain the strong metric dimension of generalized web graph with $G \sim C$ and = m without center vertex denoted by $WB_0(C_m; n)$. The generalized web graph $WB_0(C_m; n)$ can be depicted as in Figure 2.

Figure 2. Generalized web graph $WB_0(C_m; n)$

Lemma 2.4 *For every integer $m \geq 3$ and $n \geq 2$, if S is a strong resolving set of generalized web graph $WB_0(C_m; n)$ then $|S| \geq m$.*

Proof. Let S be a strong resolving set of generalized web graph $WB_0(C_m; n)$. Suppose that S contains at most m - 1 vertices, n 1 $|n$ $1|$ 1 2 $\subset$ 0 m 1 $= \{v_1^1; v_2^1; : : : ; v_m^1; v_1^2; v_2^2; : : : ; v_m^2; : : : ; v_1^{n-1}; v_2^{n-1}; : : : ; v_m^{n-1}\}$ and $V_2 = \{v_1^n; v_2^n; : : : ; v_m^n\}$. Now, we define $S_1 = V_1 \cap S$ and $S_2 = V_2 \cap S$. Without loss of generality, we may take $| S_1 | = p$, $p \geq 0$ and $| S_2 | = q$, $q > 0$. Clearly $p + q \geq m$, if not then there are two distinct vertices v_a and v_b where $v_a \in V_1 \setminus S_1$ and $v_b \in V_2 \setminus S_2$ such that for every $s \in S$ we obtain $v_a \notin I[v_b; s]$ and $v_b \notin I[v_a; s]$. This contradicts with the supposition that S is a strong resolving set. Thus $| S | \geq$ m.

Lemma 2.5 *For every integer* $m \geq 3$ *and* $n \geq 2$, *a set* $S = \{v_1^n; v_2^n; : : : ; v_m^n\}$ *is a strong resolving set of generalized web graph* $WB_0(C_m; n)$.

Proof. We prove that for every two distinct vertices $v_k^s; v_l^t \in V(WB_0(C_m; n)) \setminus S$ with $k; l \in [1, m]$ and $s; t \in [1, n\text{-}1]$, there exists a vertex $s \in S$ which strongly resolves v_k^s and v_l^t. There are three possible pairs of vertices.

(i) A pair of vertices $(v_k^s; v_l^t)$ with $k; l \in [1, m]$, $k \neq l$ and $s; t \in [1, n\text{-}1]$, $s = t$.
For every integer $k; l \in [1, m]$ without ... $\leq$ ... $\leq$ ... shortest $v_k^t - v_l^n$ path: $v_k^t; v_k^{t+1}; : : : ; v_l^t; v_l^{t+1}; : : : ; v_l^{n-1}; v_l^n$, and shortest $v_l^t - v_k^n$ path: $v_l^t; v_l^{t+1}; : : : ; v_k^t; v_k^{t+1}; : : : ; v_k^{n-1}; v_k^n$. So that $v_l^t \in I[v_k^t; v_l^n]$ and $v_k^t \in I[v_l^t; v_k^n]$.

(ii) A pair of vertices $(v_k^s; v_l^t)$ with $k; l \in [1, m]$, $k = l$ and $s; t \in [1, n\text{-}1]$, $s \neq t$.
For every integer $s; t \in [1,$ ... $\leq$ ... $\leq$ ... shortest $v_k^t - v_k^n$ path: $v_k^s; v_k^{s+1}; : : : ; v_k^t; v_k^{t+1}; : : : ; v_k^{n-1}; v_k^n$. So that $v_k^t \in I[v_k^s; v_k^n]$.

(iii) A pair of vertices $(v_k^s; v_l^t)$ with $k; l \in [1, m]$, $k \neq l$ and $s; t \in [1, n\text{-}1]$, $s \neq t$.
For every integer $k; l \in [1, m]$ and $s; t \in [1, n\text{-}1]$ without loss of generality $1 \leq k < l \leq m$ and $1 \leq s < t \leq n\text{-}1$, we obtain the shortest $v_k^s - v_l^n$ path: $v_k^s; v_k^{s+1}; : : : ; v_l^{t-1}; v_l^t; : : : ; v_l^{n-1}; v_l^n$ or $v_k^s; v_k^{s+1}; : : : ; v_l^{t-1}; v_l^t; : : : ; v_l^{n-1}; v_l^n$. So that $v_l^t \in I[v_k^s; v_l^n]$.

From every possible pairs of vertices, there exists a vertex $v_l^n \in S$ with $l \in [1, m]$ which strongly resolves $v_k^s, v_l^t \in V(WB_0(C_m; n)) \setminus S$. Thus S is a strong resolving set of generalized web graph $WB_0(C_m; n)$.

Theorem 2.2 *Let* $WB_0(C_m; n)$ *be the generalized web graph with* $m \geq 3$ *and* $n \geq 2$. *Then* $sdim(WB_0(C_m; n)) = m$.

Proof. By using Lemma 2.5, we have a set $S = \{v_1^n; v_2^n; : : : ; v_m^n\}$ is a strong resolving set of $WB_0(C_m; n)$ graph. According to Lemma 2.4, $| S | \geq m$ so that $S = \{v_1^n; v_2^n; : : : ; v_m^n\}$ is a strong metric basis of $WB_0(C_m; n)$. Hence $sdim(WB_0(C_m; n)) = m$.

3. Conclusion

According to the discussion above it can be concluded that the strong metric dimension of a lollipop graph $L_{m;n}$ and a generalized web graph $WB_0(C_m; n)$ are as stated in Theorem 2.1 and Theorem 2.2, respectively.

References

[1] **Kratica, J., V. Kovaˇcevi´-Vujˇci´c, and M. Cangalovi´c,** *Minimal Doubly Resolving Sets and The Strong Metric Dimension of Some Complex Polytope*, **Applied Mathematics and Computation** 218 *(2012), 9790–9801.*

[2] **Kratica, J., V. Kovaˇcevi´-Vujˇci´c, and M. Cangalovi´c,** *Strong Metric Dimension: A Survey*, **Yugoslav Journal of Operation Research** 24(2) *(2014), 187–198.*

[3] **Kusmayadi, T. A., S. Kuntari, D. Rahmadi, and F. A. Lathifah,** *On The Strong Metric Dimension of Some Related Wheel Graph*, **Far East Journal of Mathematical Sciences (FJMS)** 99 *(2016), no. 9, 1325–1334.*

[4] **Miller, M., O. Phanalasy, and J. Ryan,** *Antimagicness of Some Families of Generalized Graphs*, **Australasian Journal of Combinatorics** 53 *(2012), 179–190.*

[5] Oelermann, O. and J. Peters-Fransen, *The Strong Metric Dimension of Graphs and Digraphs****, Discrete Applied Mathematics*** 155 ***(2007), 356–364.***

[6] Seb¨o, A. and E. Tannier, *On Metric Generators of Graph****, Mathematics and Operations Research*** 29(2) ***(2004), 383–393.***

[7] Weisstein, Eric W., *CRC Concise Encyclopedia of Mathematics CD-ROM****, 2nd ed., CRC Press, Boca Raton, 2003.***

[8] Yi, E., *On Strong Metric Dimension Graphs and Their Complements****, Acta Mathematica Sinica*** 29(8) ***(2013), 1479–1492.***