On The Strong Metric Dimension of Lollipop Graph and Generalized Web Graph

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Abstract. Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The interval $I[u; v]$ between $u$ and $v$ to be the collection of all vertices that belong to some shortest $u - v$ path. A vertex $s$ strongly resolves two vertices $u$ and $v$ if $u$ belongs to a shortest $v - s$ path, denoted by $u 2 I[v; s]$ or $v$ belongs to a shortest $u - s$ path, denoted by $v 2 I[u; s]$. A vertex set $S$ of $G$ is a strong resolving set of $G$ if every two distinct vertices of $G$ are strongly resolved by some vertex of $S$. The strong metric basis of $G$ is a strong resolving set with minimal cardinality. The strong metric dimension $sdim(G)$ of a graph $G$ is defined as the cardinality of strong metric basis. In this paper we determine the strong metric dimension of a lollipop graph and a generalized web graph. Lollipop graph $L_{m;n}$ is the graph obtained by joining a complete graph $K_m$ to a path graph $P_n$ with a bridge. We obtain the strong metric dimension of a lollipop graph $L_{m;n}$ is $m + 1$. Generalized web graph $W(B(G; m; n))$ is the graph obtained from the generalized pyramid graph $P(G; m)$ by taking $p$ copies of $P_n$ and merging an end vertex of a different copy of $P_n$ with each vertex of the farthest copy of $G$ from the apex. We obtain the strong metric dimension of generalized web graph with $G$ and without center vertex is $m + n$.

1. Introduction

Let $G$ be a simple connected undirected graph $G = (V; E)$, where $V$ is a set of vertices, and $E$ is a set of edges. The distance between vertices $u$ and $v$, i.e. the length of a shortest $u - v$ path is denoted by $d(u; v)$. A vertex set $B = \{x_1; x_2; \ldots ; x_k\}$ of $G$ is a resolving set of $G$ if every two distinct vertices of $G$ are resolved by some vertex of $B$. The metric basis of $G$ is a resolving set with minimal cardinality. The metric dimension of $G$, denoted by $dim(G)$ is the cardinality of its metric basis.

The concept of strong metric dimension was introduced by Seb˝o and Tannier [6] in 2004. Kratica et al. [2] defined for two vertices $u$ and $v$ in a connected graph $G$, if $u$ belongs to a shortest $v - s$ path, denoted by $u 2 I[v; s]$ or $v$ belongs to a shortest $u - s$ path, denoted by $v 2 I[u; s]$ then a vertex $s$ strongly resolves two vertices $u$ and $v$. A set of vertices $S \subseteq V(G)$ is a strong resolving set for $G$ if every two distinct vertices $u, v 2 V(G)$ are strongly resolved by some vertex of $S$. The strong metric dimension of $G$ denoted by $sdim(G)$ is the minimum cardinality over all strong resolving sets of $G$.

Many researchers have investigated the strong metric dimension to some graph classes. In 2004 Seb˝o and Tannier [6] observed that the strong metric dimension of complete graph $K_n$ is $n - 1$, cycle graph $C_n$ is $\lceil n/2 \rceil$, and tree is $L(T) - 1$, where $L(T)$ denotes the number of leaves.
of tree. In 2013 Yi [8] determined that the strong metric dimension of $G$ is 1 if and only if $G = P_n$.
Kusmayadi et al. [3] determined the strong metric dimension of some related wheel graph such as sunflower graph, $r$-fold wheel graph, helm graph, and friendship graph. In this paper, we determine the strong metric dimension of a lollipop $L_{m;n}$ graph and a generalized web $WB(G; m; n)$ graph.

2. Main Result
2.1. Strong Metric Dimension

Let $G$ be a connected graph with vertex set $V(G)$, edge set $E(G)$, and $S = \{s_1; s_2; \ldots ; s_k\} \subseteq V(G)$. Oellermann and Peters-Fransen [5] defined the interval $I[u; v]$ between $u$ and $v$ to be the collection of all vertices that belong to some shortest $u - v$ path. A vertex $s \in S$ strongly resolves two vertices $u$ and $v$ if $u \in I[v; s]$ or $v \in I[u; s]$. A vertex set $S$ of $G$ is a strong resolving set of $G$ if every two distinct vertices of $G$ are strongly resolved by some vertex of $S$. The strong metric basis of $G$ is a strong resolving set with minimal cardinality. The strong metric dimension of a graph $G$ is defined as the cardinality of strong metric basis denoted by $sdim(G)$. We often make use of the following lemma and properties about strong metric dimension given by Kratica et al. [1].

Lemma 2.1 Let $u, v \in V(G)$, $u \neq v$, and

(i) $d(w,v) \leq d(u,v)$ for each $w$ such that $\{w,u\} \in E(G)$, and
(ii) $d(u,w) \leq d(u,v)$ for each $w$ such that $\{v,w\} \in E(G)$.

Then, there does not exist vertex $a \in V(G)$, $a \neq u,v$ that strongly resolves vertices $u$ and $v$.

![Figure 1. Lollipop graph $L_{m;n}$](image)

Lemma 2.2 For every integer $m \geq 3$ and $n \geq 1$, if $S$ is a strong resolving set of lollipop graph $L_{m;n}$ then $|S| \geq m - 1$. 

The strong metric dimension of $L_{m;n}$ graph and a generalized web $WB(G; m; n)$ graph.
Proof. Consider a pair of vertex \((v_i; u_n)\) with \(i \in [1, m-1]\) satisfying condition \(d(v_i; u_n) = n+1 = diam(L_{m;n})\). According to Property 2.2, we obtain \(v_i \in S\) or \(u_n \in S\), therefore \(S\) has at least \(m - 1\) vertices. Hence \(|S| \geq m - 1\).

Lemma 2.3 For every integer \(m \geq 3\) and \(n \geq 1\), a set \(S = \{v_1; v_2; \ldots; v_m\}\) is a strong resolving set of lollipop graph \(L_{m;n}\).

Proof. For every integer \(i \in [1, m-1]\) satisfying condition \(d(u_n; v_i) = n+1 = diam(L_{m;n})\), we obtain the shortest \(u_n - v_i\) path: \(u_n; u_n 1; u_n 2; \ldots; u_i; v_m; v_i\). So that \(v_i\) strongly resolves a pair of vertices \((v_m; u_n)\). Thus \(v_m \in I[u_n; v_i]\).

For a pair of vertices \((u_k; u_l)\) with \(k \neq l, k; l \in [1, n-1]\), without loss of generality \(k < l\) we obtain for \(i \in [1, m-1]\), the shortest \(u_l - v_i\) path: \(u_l; u_l 1; \ldots; u_k; u_k 1; \ldots; u_i; v_m; v_i\). So that \(v_i\) strongly resolves a pair of vertices \((u_k; u_l)\). Thus \(u_k \in I[u_l; v_i]\).

Therefore \(S = \{v_1; v_2; \ldots; v_m\}\) is a strong resolving set of lollipop graph \(L_{m;n}\).

Theorem 2.1 Let \(L_{m;n}\) be the lollipop graph with \(m \geq 3\) and \(n \geq 1\). Then \(sdim(L_{m;n}) = m - 1\).

Proof. By using Lemma 2.3 a set \(S = \{v_1; v_2; \ldots; v_m\}\) is a strong resolving set of lollipop graph \(L_{m;n}\) with \(m \geq 3\) and \(n \geq 1\). According to Lemma 2.2, \(|S| \geq m - 1\) so that \(S\) is a strong metric basis of lollipop graph \(L_{m;n}\). Hence \(sdim(L_{m;n}) = m - 1\).

2.3. The Strong Metric Dimension of Generalized Web Graph

Miller et al. [4] defined the generalized web graph \(WB(G; m; n)\) for \(m \geq 3\), \(n \geq 2\) is the graph obtained from the generalized pyramid \(P(G; m)\) by taking \(p\) copies of \(P_n\) \((n \geq 2)\) and merging an end vertex of a different copy of \(P_n\) with each vertex of the furthermost copy of \(G\) from the apex.

In this paper we obtain the strong metric dimension of generalized web graph with \(G \sim C\) and \(n = m\) without center vertex denoted by \(WB_0(C_m; n)\). The generalized web graph \(WB_0(C_m; n)\) can be depicted as in Figure 2.

Figure 2. Generalized web graph \(WB_0(C_m; n)\)

Lemma 2.4 For every integer \(m \geq 3\) and \(n \geq 2\), if \(S\) is a strong resolving set of generalized web graph \(WB_0(C_m; n)\) then \(|S| \geq m\).
Proof. Let \( S \) be a strong resolving set of a generalized web graph \( W B_0(C_m; n) \). Suppose that \( S \) contains at most \( m - 1 \) vertices, \( n \leq 1 \)

\[
W B_0(C_m; n) = \{v_1; v_2 ; \ldots ; v_m \} \quad \text{and} \quad V_2 = \{v_1; v_2 ; \ldots ; v_m \}. \]

Now, we define \( S_1 = V_1 \cap S \) and \( S_2 = V_2 \cap S \). Without loss of generality, we may take \( |S_1| = p \), \( p \geq 0 \) and \( |S_2| = q \), \( q > 0 \). Clearly \( p + q \geq m \), if not then there are two distinct vertices \( v_q \) and \( v_q \) where \( v_q \in V_1 \setminus S_1 \) and \( v_q \in V_2 \setminus S_2 \) such that for every \( s \in S \) we obtain \( v_q \in I[v_q; s] \) and \( v_q \in I[v_q; s] \). This contradicts with the supposition that \( S \) is a strong resolving set. Thus \( |S| \geq m \).

Lemma 2.5 For every integer \( m \geq 3 \) and \( n \geq 2 \), a set \( S = \{v_1; v_2 ; \ldots ; v_m \} \) is a strong resolving set of a generalized web graph \( W B_0(C_m; n) \).

Proof. If \( S \) is a strong resolving set of \( W B_0(C_m; n) \), then \( v_k \in S \) and \( v_l \in S \). There are three possible pairs of vertices.

(i) A pair of vertices \( (v_k; v_l) \) with \( k \in [1, m] \) and \( l \in [1, n-1] \), \( s \in S \).

For every integer \( k, l \in [1, m] \) without loss of generality, \( n \leq 1 \) and \( \leq 1 \), shortest \( v_k \) path: \( v_k \), \( v_k \), \( v_k \), \( v_k \), \( v_k \), \( v_k \), and shortest \( v_l \) path: \( v_l \), \( v_l \), \( v_l \), \( v_l \), \( v_l \), \( v_l \). So that \( v_l \in I[v_k; v_l] \) and \( v_k \in I[v_l; v_k] \).

(ii) A pair of vertices \( (v_k; v_l) \) with \( k \in [1, m] \), \( l \in [1, n] \) and \( s \in S \).

For every integer \( s, t \in [1, m] \), \( s \leq t \), shortest \( v_k \) path: \( v_k \), \( v_k \), \( v_k \), \( v_k \), \( v_k \), \( v_k \). So that \( v_k \in I[v_k; v_k] \).

(iii) A pair of vertices \( (v_k; v_l) \) with \( k \in [1, m] \), \( s \leq t \), shortest \( v_k \) path: \( v_k \), \( v_k \), \( v_k \), \( v_k \), \( v_k \), \( v_k \). So that \( v_k \in I[v_k; v_k] \).

From every possible pairs of vertices, there exists a vertex \( v_l \in S \) with \( l \in [1, m] \) which strongly resolves \( v_k \). Thus \( S \) is a strong resolving set of a generalized web graph \( W B_0(C_m; n) \).

Theorem 2.2 Let \( W B_0(C_m; n) \) be the generalized web graph with \( m \geq 3 \) and \( n \geq 2 \). Then \( sdim(W B_0(C_m; n)) = m \).

Proof. By using Lemma 2.5, we have a set \( S = \{v_1; v_2 ; \ldots ; v_m \} \) is a strong resolving set of \( W B_0(C_m; n) \) graph. According to Lemma 2.4, \( |S| \geq m \) so that \( S = \{v_1; v_2 ; \ldots ; v_m \} \) is a strong metric basis of \( W B_0(C_m; n) \). Hence \( sdim(W B_0(C_m; n)) = m \).

3. Conclusion

According to the discussion above it can be concluded that the strong metric dimension of a lollipop graph \( L_{m;n} \) and a generalized web graph \( W B_0(C_m; n) \) are as stated in Theorem 2.1 and Theorem 2.2, respectively.

References


