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A comparative study of numerical methods for estimating the relationship between cosmic energy and the expansion rate of the universe

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Abstract: This research aims to develop a numerical method that can accurately estimate the relationship between cosmic energy (E) and the expansion rate of the universe (H), taking into account the complex interactions between ordinary matter, dark matter, and dark energy. Numerical approaches based on Euler, Runge-Kutta, and Adams-Bashforth integration methods will be refined to evaluate the correlation. The limitation of this study is to a flat universe (k = 0 geometry), but it has the potential to be extended to other geometries. This effective numerical method can revolutionize cosmology by allowing accurate testing of cosmological theories and improving predictive capabilities. This study not only deepens our understanding of the behavior of the universe, but also opens up opportunities for further exploration. While there has been research on the Friedmann equation and the evolution of the universe, this study fills the gap by comparing three numerical methods, promising a more comprehensive and accurate analysis. This research demonstrates significant advances in cosmological methodology, with the potential to change the cosmological paradigm through efficient numerical approaches. By improving the understanding of cosmic energy and the expansion rate of the universe, this research not only contributes to the current knowledge of cosmology, but also paves the way for impactful follow-up research in this field.

Keyword: Numerical method; Cosmic energy; Expansion rate; Cosmology; Dark matter.

1. Introduction

In previous cosmological research, scientists have examined the role of the Friedmann equation in modeling the evolution of the universe (Chavanis, 2014; Nemiroff & Patla, 2008; Ren & Meng, 2006). This equation, first introduced by Alexander Friedmann in 1922, provides a mathematical framework for understanding how parameters such as the expansion rate of the universe (represented by the Hubble function) are related to various important components such as ordinary matter, dark matter, dark energy, and the geometry of the universe (Friedmann, 2014; Klimchitskaya & Mostepanenko, 2022). Previous research, such as that conducted by Edwin Hubble in 1929 with observations of galaxies, has provided evidence that the universe is expanding (Bahcall, 2015; Kirshner, 2004). Through the Friedmann equation, we can understand the evolution of the universe from the Big Bang to the present day (Carroll & Kaplinghat, 2002). However, to explore the role of each of these components in the evolution of the universe, numerical methods are needed that can estimate their values over time, as highlighted by recent research in cosmology.

This research aims to develop a numerical method that can estimate the relationship between cosmic energy (E) and the expansion rate of the universe (H) with greater accuracy in physics. Through this research, we hope to gain a deeper insight into how various components, such as ordinary matter, dark matter and dark energy, affect the development of the universe. This will improve our understanding of the dynamics of the Universe and provide a solid foundation for further research in cosmology.

This research will focus on improving three numerical approaches to project the correlation between cosmic energy (E) and the expansion rate of the universe (H), namely by using the Euler integration method, the Runge-Kutta method, and the Adams-Bashforth method (Biswas et al., 2013; Butcher, 2007; Durran, 1991). This research will be limited to the context of a flat universe ($k = 0$ geometry) to simplify the analysis, although this approach has the potential to be extended to consider diverse geometries.

The development of effective numerical methods for modeling the evolution of the universe can have far-reaching impacts in the field of cosmology. With a more accurate and efficient approach, this research can help cosmologists to test and verify existing theories, as well as better predict the behavior of the universe. It can also open the door for further research in understanding the nature of the universe. The development of effective numerical methods to model the evolution of the universe has the potential to change the paradigm in cosmology. With a more precise and efficient approach, this research allows cosmologists to test and confirm existing theories, as well as improve their ability to predict the behavior of the universe. The application of this method also paves the way for further research in deepening our understanding of the nature of the universe.

Although many studies have been conducted on the Friedmann equation and the evolution of the universe, there is still a need to develop more accurate and efficient numerical methods to estimate the relationship between cosmic energy and the expansion rate of the universe. Some studies have neglected some important aspects or used inappropriate approaches, so there are gaps in the research that can be filled by this study.

This research is unique in that it involves the development and comparison of three different numerical methods for estimating the relationship between cosmic energy and the expansion rate of the universe. This approach allows for a more in-depth analysis of the dynamics of the universe and produces more accurate results compared to previous methods. Therefore, this research has the potential to be an important contribution to the field of cosmology.

2. Research Method

2.1. Friedmann Equation in Cosmology

Knowing that (H_0) is the Hubble function at the present time (t_0), we can write the Friedmann equation as follows in (1):

$$H_0^2 = \frac{8\pi G}{3}\rho_0 - \frac{k}{a_0^2} + \frac{\Lambda}{3} \tag{1}$$

In physics, (H_0) is a parameter known as the Hubble current parameter. It indicates the rate at which the universe is expanding at the moment. The larger the value of (H_0), the faster our universe is expanding. (G) is Newton's gravitational constant, a fundamental value in physics, which is responsible for the gravitational force between objects in the universe. (ρ_0) is the average energy density of the universe at this point in time. It includes all forms of energy, including ordinary matter, dark matter, and dark energy (Turner, 2000). (k) is a parameter that determines the geometry of the universe. Positive values indicate a closed universe (spherical geometry), negative values indicate an open universe (hyperbola geometry), and zero values indicate a flat universe (Cornish & Weeks, 1998). (a_0) is the scale factor of the universe at this point in time, which indicates how much the universe has expanded since the Big Bang relative to its size at that time. (Λ) is the cosmological constant, also known as dark energy or the cosmological constant. This constant describes the acceleration of the expansion of the universe. Note that $\{\rho(t)\}$ has the form (ρ_0), which is the energy density at the present time:

$$\rho(t) = \rho_0 \left\{\frac{a_0}{a(t)}\right\}^3 \tag{2}$$

substitution $\{\rho(t)\}$ into Friedmann equation:

$$H^2(t) = \frac{8\pi G}{3}\rho_0 \left\{\frac{a_0}{a(t)}\right\}^3 - \frac{k}{a^2(t)} + \frac{\Lambda}{3} \tag{3}$$

From the above equation, we can divide both sides of the equation by (H_0^2):

$$\frac{H^2(t)}{H_0^2} = \frac{\frac{8\pi G}{3}\rho_0 \left\{\frac{a_0}{a(t)}\right\}^3 - \frac{k}{a^2(t)} + \frac{\Lambda}{3}}{H_0^2} \tag{4}$$

This equation divides the square of the Hubble parameter at time (t) by the square of the current Hubble parameter (H_0), providing an understanding of the expansion rate of the universe at time (t) relative to the current expansion rate (Moresco et al., 2012). On the top right, there is a term that reflects the contribution of the average energy density of the universe (ρ_0), which includes various forms of energy such as ordinary matter, dark matter and dark energy (Sahni, 2004). This term is affected by the scale factor of the universe $\{a(t)\}$, indicating how much the universe has grown since the Big Bang. The larger $\{a(t)\}$ is relative to $\{a_0\}$, the greater the contribution of this term to the expansion rate of the universe. On the bottom right, there is a term related to the geometry parameter of the universe (k), determining whether the universe is closed (k > 0), open (k < 0), or flat (k = 0) (Melia & Shevchuk, 2012). This term changes over time due to changes in $\{a(t)\}$. There is also a term related to the cosmological constant (Λ), representing the dark energy or cosmological constant that affects the acceleration or deceleration of the expansion of the universe (S. Turner & Huterer, 2007):

$$\Omega_m = \frac{8\pi G\rho_0}{3H_0^2} \tag{5}$$

$$\Omega_k = -\frac{k}{a_0^2 H_0^2} \tag{6}$$

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2} \tag{7}$$

These physical equations provide a mathematical interpretation of the relative roles of various components in the universe to the evolution and geometry of the universe. The equation for the matter density parameter (Ω_m) is obtained by dividing $8\pi G\rho_0$ by $3H_0^2$, where G is the gravitational constant, ρ_0 is the average energy density of the universe at this time, and H_0 is the current Hubble parameter. The interpretation is that the larger the value of (Ω_m), the greater the contribution of matter to the total energy in the universe, resulting in a slowdown in the expansion rate of the universe (Frautschi, 1982). The equation for the critical density parameter (Ω_k) is calculated by dividing -k by $(a_0^2 H_0^2)$, where k is a parameter of the geometry of the universe that determines whether the universe is closed (k > 0), open (k < 0), or flat (k = 0). The interpretation is that the value of (Ω_k) indicates how far the geometry of the universe differs from a flat state, with positive values indicating a closed universe, negative values indicating an open universe, and zero values indicating a flat universe (Bahcall et al., 1999). The equation for the dark energy parameter (Ω_Λ) is obtained by dividing Λ by $3H_0^2$, where Λ is the cosmological constant. The interpretation is that the larger the value of (Ω_Λ), the greater the contribution of dark energy or cosmological constant to the total energy in the universe, which causes the accelerated expansion of the universe (Frieman et al., 2008):

$$\frac{H^2(t)}{H_0^2} = \Omega_m \left(\frac{a_0}{a(t)}\right)^3 + \Omega_k \left(\frac{a_0}{a(t)}\right)^2 + \Omega_\Lambda \tag{8}$$

The term $\Omega_m \{a_0 / a(t)\}^3$ describes the contribution of matter to the overall energy of the universe, with Ω_m as an index of the density of matter. The factor $\{a_0 / a(t)\}^3$ reflects the decrease in matter density as space expands. The term $\Omega_k \{a_0 / a(t)\}^2$

represents the impact of geometry on the total energy, governed by Ω_k as the critical density parameter. The factor $\{a_0 / a(t)\}^2$ evaluates the effect of geometry on the space-time of the universe. The term Ω_Λ reflects the role of dark energy or cosmological constant in the overall energy of the universe. The value of Ω_Λ describes the degree of dark energy contribution to the accelerated expansion of the universe. This equation is similar to the general form of the Friedmann equation. Rewritten as $H(z)^2 = \left(8\pi G\rho_0 / 3H_0^2\right)(1+z)^3 + \left(16\pi^2 G^2 \rho_0^2 / 3\Lambda a_0^2 H_0^2\right)(1+z)^2 + \left(\Lambda / 3H_0^2\right)$, where (z) is the redshift, we get the equation:

$$\frac{H^2(z)}{H_0^2} = \Omega_m (1+z)^3 + \Omega_k (1+z)^2 + \Omega_\Lambda \tag{9}$$

This equation is the generalized form of the Friedmann equation in cosmology. $\left\{H^2(z)/H_0^2\right\}$ is the ratio of the square of the Hubble function at redshift (z) to the squared value of the Hubble function at the present time (H_0). $\left\{\Omega_m(1+z)^3\right\}$ is the contribution of ordinary matter (dark matter and stars) in the universe. As the universe expands, the density of this matter decreases, but the volume of the universe $\left\{(1+z)^3\right\}$ also increases, so the contribution of this material depends on cosmological parameters (Ω_m) and scale factor $\left\{(1+z)^3\right\}$. $\left\{\Omega_k(1+z)^2\right\}$ is the contribution from the cosmological balance (radiation or dark matter). (Ω_k) is a parameter that determines the geometry of the universe (flat, closed, or open). Scale factor $\left\{(1+z)^2\right\}$ include the effect of redshift on cosmological balance. The equation states that the rate of expansion of the universe at redshift (z) depends on the contributions of ordinary matter, cosmological balance and dark energy, each of which is determined by the cosmological parameter (Ω_m), (Ω_k), and (Ω_Λ).

2.2. Approximate Relationship Between Cosmic Energy (E) and Universe Expansion Rate (H) Through Taylor Series Approach

To obtain an approximate relationship between (E) and (H), we will make some assumptions and approximations. We will assume that at this moment (z=0), the value of the Hubble function is (H_0), and the energy function is (E0). We will use the Taylor approximation to expand the Friedmann equation around (z=0). The given equation is the Taylor expansion of the function (E(z)) around the point (z=0), which is the initial approach to approximating (E(z)) with a Taylor series up to second order (Baeza et al., 2017). We can approximate the relationship between cosmic energy (E) and the expansion rate of the universe (H) by expanding (E(z)) in Taylor series around the point (z=0). Starting with a reminder of the Taylor expansion formula (Pourahmadi, 1984):

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \tag{10}$$

Define remainder $\{R_n(x)\}$ of the Taylor expansion:

$$R_n(x) = f(x) - \left\{ f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots \right\} \tag{11}$$

$\{R_n(x)\}$ approaches zero as (n) approaches infinity. Use the remainder formula for the Taylor expansion (Poffald, 1990):

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{12}$$

where (c) is between (x) and (a). This is known as the Peano remainder form of the Taylor remainder. Prove that $(R_n(x) \to 0)$ at $(n \to \infty)$. To prove this, we will use the Mean Value Theorem (MVT) (Smoryński, 2017). Suppose $(x > a)$, then there is a (c) between (a) and (x) such that:

$$f^{(n+1)}(c) = \frac{f^{(n+1)}(c)}{1!} \tag{13}$$

Then:

$$|R_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \right| \tag{14}$$

Because (x > a), so $(|x - a| > 0)$, and since (c) remains between (a) and (x), then $(|f^{(n+1)}(c)|)$ is constant. So,

$$|R_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} \right| |x-a|^{n+1} \tag{15}$$

$\left| \frac{f^{(n+1)}(c)}{(n+1)!} \right|$ is constant because (c) is fixed. So,

$$|R_n(x)| = \text{constan} \times |x-a|^{n+1} \tag{16}$$

When $(n \to \infty)$, $(|x-a|^{n+1})$ will be close to zero because $(|x - a| < 1)$ (because (x) is greater than (a)). Thus, $\{R_n(x) \to 0\}$ at $(n \to \infty)$. This proves that the remainder of the Taylor expansion approaches zero when $(n \to \infty)$, which means that the Taylor expansion approaches the original function when $(n \to \infty)$, corresponds to what is stated in the Taylor expansion formula (Makino & Berz, 2003). We will use the Taylor series to approximate the function (E(z)) around the point (z = 0). The Taylor series of a function (f(x)) around the point (x = a) is given by:

$$f(x) = f(a) + \frac{df}{dx}\Big|_{x=a}(x-a) + \frac{1}{2}\frac{d^2 f}{dx^2}\Big|_{x=a}(x-a)^2 + \ldots \tag{17}$$

We replace (x) with (z) and (a) with (0), so the Taylor series for (E(z)) around (z = 0) is:

$$E(z) = E(0) + \frac{dE}{dz}\Big|_{z=0} z + \frac{1}{2}\frac{d^2 E}{dz^2}\Big|_{z=0} z^2 + \ldots \tag{18}$$

To get an approximate equation $\left\{H^2(z)/H_0^2 = \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda\right\}$ by using the Taylor expansion of the function (E(z)), we need to first expand the function to a sufficient order. We need to calculate the first and second derivatives of (E(z)) with respect to (z) at (z = 0) to obtain the corresponding coefficients. First derivative:

$$E'(z) = \frac{dE}{dz} = \frac{d}{dz}\left\{ E(0) + \frac{dE}{dz}|_{z=0}\ z + \frac{1}{2}\frac{d^2E}{dz^2}|_{z=0}\ z^2 + \ldots \right\} \tag{19}$$

$$= \frac{dE}{dz}|_{z=0} + \frac{d^2E}{dz^2}|_{z=0}\ z + \ldots$$

We evaluate this first derivative at (z = 0), so that:

$$E'(0) = \frac{dE}{dz}|_{z=0} \tag{20}$$

$$= \frac{dE}{dz}|_{z=0} + \frac{d^2E}{dz^2}|_{z=0}\ z + \ldots$$

Second derivative:

$$E''(z) = \frac{d^2E}{dz^2} = \frac{d}{dz}\left\{ \frac{dE}{dz}|_{z=0} + \frac{d^2E}{dz^2}|_{z=0}\ z + \ldots \right\} \tag{21}$$

$$= \frac{d^2E}{dz^2}|_{z=0} + \ldots$$

We evaluate this second derivative at (z = 0), so that:

$$E''(0) = \frac{d^2E}{dz^2}|_{z=0} \tag{22}$$

After obtaining the coefficients of the first and second derivatives, we can substitute them into the Taylor expansion of the function (E(z)). Then, using the Friedmann-Lemaitre-Robertson-Walker (FLRW) equation for a flat universe, we will approximate the change in the expansion rate, (H(z)):

$$H(z) = H_0 E(z) \tag{23}$$

By substituting the found values into the equation $\left\{ H^2(z)/H_0^2 = \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda \right\}$, we can approximate the dynamics of the universe. In the context of the Taylor expansion, we get a more detailed picture of the evolution of the Hubble parameter (H(z)), defined as $H(z) \approx H_0\left\{ E(0) + E'(0)z + 1/2 \cdot E''(0)z^2 \right\}$. Therefore,

$$H^2(z) \approx H_0^2[E^2(0) + 2E(0)E'(0)z + E'(0)^2 z^2] \tag{24}$$

Substituting it into the FLRW equation, we get:

$$\Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda \approx \frac{H^2(z)}{H_0^2} \approx E^2(0) + 2E(0)E'(0)z + E'(0)^2 z^2 \tag{25}$$

Substitution Ω_m, Ω_k, and Ω_Λ, we get:

$$E^2(0) + 2E(0)E'(0)z + E'(0)^2 z^2 \approx \frac{8\pi G\rho_0}{3H_0^2}(1+z)^3 - \frac{k}{a_0^2 H_0^2}(1+z)^2 + \frac{\Lambda}{3H_0^2} \tag{26}$$

By comparing the coefficients on both sides of the equation, we can obtain approximate values of (E(0)), (E'(0)), and (E''(0)). We will use this approach to estimate the desired equation. We analyze the coefficients on both sides of the equation. On the left side of the equation, we have:

$$E^2(0)+2E(0)E'(0)z+E'(0)^2 z^2 \tag{27}$$

Coefficient for (z^0) is $(E^2(0))$, for (z^1) is $\{2E(0)E'(0)\}$, and for (z^2) is $(E'(0)^2)$. On the right side of the equation, we have:

$$\frac{8\pi G\rho_0}{3H_0^2}(1+z)^3-\frac{k}{a_0^2H_0^2}(1+z)^2+\frac{\Lambda}{3H_0^2} \tag{28}$$

Coefficient for (z^0) is $(\Lambda/3H_0^2)$, for (z^1) is $(8\pi G\rho_0/3H_0^2)$, and for (z^2) is $(-k/a_0^2H_0^2)$. We can balance the coefficients for each order (z) by comparing them. Then we will have a system of equations to determine (E(0)), (E'(0)), and (E"(0)).

$$E^2(0)=\frac{\Lambda}{3H_0^2} \tag{29}$$

$$2E(0)E'(0)=\frac{8\pi G\rho_0}{3H_0^2} \tag{30}$$

$$E'(0)^2=-\frac{k}{a_0^2H_0^2} \tag{31}$$

From equation, we have:

$$E^2(0)=\frac{\Lambda}{3H_0^2} \tag{32}$$

Since we want to find (E(0)), we can take the square root of both sides of the equation:

$$E(0)=\sqrt{\frac{\Lambda}{3H_0^2}} \tag{33}$$

From equation, we have:

$$2E(0)E'(0)=\frac{8\pi G\rho_0}{3H_0^2} \tag{34}$$

We already know the value of (E(0)) from the previous step, so we can substitute it into the equation:

$$2\sqrt{\frac{\Lambda}{3H_0^2}}E'(0)=\frac{8\pi G\rho_0}{3H_0^2} \tag{35}$$

Then we solve for (E'(0)):

$$E'(0)=\frac{4\pi G\rho_0}{\sqrt{3\Lambda}}=\frac{4\pi G\rho_0}{\sqrt{3}\Lambda^{1/2}} \tag{36}$$

From equation, we have:

$$E'(0)^2=-\frac{k}{a_0^2H_0^2} \tag{37}$$

We already know the value of (E'(0)) from the previous step, so we can substitute it into the equation:

$$\left(\frac{4\pi G\rho_0}{\sqrt{3}\Lambda^{1/2}}\right)^2=-\frac{k}{a_0^2H_0^2} \tag{38}$$

Then we solve for (k):

$$k = -\frac{16\pi^2 G^2 \rho_0^2}{3\Lambda} \tag{39}$$

2.3. Model matematika numerik

2.3.1. Euler integration method

To use Euler's numerical integration method to solve such differential equations, the first step is to convert it into an integrable ordinary differential equation (ODE) (Cryer & Tavernini, 1972). This is done by dividing both sides of the equation by $\left(H_0^2/3\right)$ and replacing the variable $\{y = H(z)/H_0\}$. We get the differential equation:

$$y^2(z) = \frac{8\pi G\rho_0}{3}(1+z)^3 + \frac{16\pi^2 G^2 \rho_0^2}{3\Lambda a_0^2}(1+z)^2 + \frac{\Lambda}{3} \tag{40}$$

We have a differential equation $\{dy/dz = f(z, y)\}$, where (f(z, y)) is a given function. To solve this differential equation, we will apply Euler's numerical integration method. We choose initial conditions, for example (z0 = 0) and $\{y(z_0) = 1\}$, because we want to solve for (H(z)) relative to (H0), so that at (z = 0), ($H(z) = H_0$). Then, we choose a small step (h) to perform numerical integration. The iterative Euler formula used is:

$$y_{n+1} = y_n + hf(z_n, y_n) \tag{41}$$

Where (y_n) is the numerical solution at step (n), (z_n) is the value of (z) at step (n), and (h) is the chosen integration step. We can calculate the numerical solution (y) iteratively by updating the value of (y) at each step using the derivative {f(z, y)} at the previous step:

$$y_{i+1} = y_i + h \cdot f(z_i, y_i) \tag{42}$$

With this formula, we can calculate the (y) value at point (z_{i+1}) based on the (y) value at point (z_i). This process is repeated until reaching the desired (z) value. After obtaining a series of (z) and (y) values, we can plot a result of {y(z)} to see how {H(z)} evolves along with (z). Using Euler's numerical integration method, we can calculate the value of {H(z)} relative to (H_0) for various values of (z) according to the given differential equation**.**

2.3.2. Runge-kutta Method

To solve this differential equation utilizing the Runge-Kutta method, we must first reconfigure it to resemble an ordinary differential equation (ODE) (Cryer & Tavernini, 1972). The equation is the Friedmann equation, which delineates the universe's scale evolution in cosmology. To better align with conventional cosmological notation, we redefine the equation in terms of $\{H(z) = \dot{a}(z)/a(z)\}$, where $a(z)$ denotes the scale factor at time (t) relative to the redshift (z), and $\{\dot{a}(z)\}$ represents the time derivative of the scale factor concerning the redshift. Utilizing this correlation, the Friedmann equation can be reformulated into the structure of an ordinary differential equation as follows:

$$\left(\frac{\dot{a}(z)}{a(z)}\right)^2 = \frac{8\pi G\rho_0}{3}\frac{1}{H_0^2}(1+z)^3 + \frac{16\pi^2 G^2 \rho_0^2}{3\Lambda a_0^2 H_0^2}(1+z)^2 + \frac{\Lambda}{3H_0^2} \tag{43}$$

Define $(x = a)$ and $(y = \dot{a})$, so that we have a system of first-order differential equations:

$$\begin{cases} \dfrac{dx}{dz} = y \\ \dfrac{dy}{dz} = \left(\dfrac{8\pi G\rho_0}{3}\dfrac{1}{H_0^2}(1+z)^3 + \dfrac{16\pi^2 G^2 \rho_0^2}{3\Lambda a_0^2 H_0^2}(1+z)^2 + \dfrac{\Lambda}{3H_0^2}\right)x - x^2 \end{cases} \tag{44}$$

Now we can use the Runge-Kutta method of order 4 (RK4) to solve this system of ordinary differential equations numerically (Islam, 2015). RK4 is a commonly used numerical method for solving ordinary differential equations (Bakir & Mert, 2022). To start, we need to calculate (k_1) and (l_1) using the given equations:

$$k_1 = y \tag{45}$$

$$l_1 = \left(\frac{8\pi G\rho_0}{3}\frac{1}{H_0^2}(1+z)^3 + \frac{16\pi^2 G^2 \rho_0^2}{3\Lambda a_0^2 H_0^2}(1+z)^2 + \frac{\Lambda}{3H_0^2}\right)x - x^2 \tag{46}$$

Calculate (k_2) and (l_2) using the (x) and (y) values generated from step 1:

$$k_2 = y + \frac{h}{2}l_1 \tag{47}$$

$$l_2 = \left[\left\{\frac{8\pi G\rho_0}{3}\frac{1}{H_0^2}\left(1+z+\frac{h}{2}\right)^3 + \frac{16\pi^2 G^2 \rho_0^2}{3\Lambda a_0^2 H_0^2}\left(1+z+\frac{h}{2}\right)^2 + \frac{\Lambda}{3H_0^2}\right\}\left(x+\frac{h}{2}k_1\right)\right] - \left(x+\frac{h}{2}k_1\right)^2 \tag{48}$$

Calculate (k_3) and (l_3) using the (x) and (y) values resulting from step 2:

$$k_3 = y + \frac{h}{2}l_2 \tag{49}$$

$$l_3 = \left[\left\{\frac{8\pi G\rho_0}{3}\frac{1}{H_0^2}\left(1+z+\frac{h}{2}\right)^3 + \frac{16\pi^2 G^2 \rho_0^2}{3\Lambda a_0^2 H_0^2}\left(1+z+\frac{h}{2}\right)^2 + \frac{\Lambda}{3H_0^2}\right\}\left(x+\frac{h}{2}k_2\right)\right] - \left(x+\frac{h}{2}k_2\right)^2 \tag{50}$$

Calculate (k_4) and (l_4) using the (x) and (y) values resulting from step 3:

$$k_4 = y + hl_3 \tag{51}$$

$$l_4 = \left(\frac{8\pi G\rho_0}{3}\frac{1}{H_0^2}(1+z+h)^3 + \frac{16\pi^2 G^2 \rho_0^2}{3\Lambda a_0^2 H_0^2}(1+z+h)^2 + \frac{\Lambda}{3H_0^2}\right)(x+hk_3) - (x+hk_3)^2 \tag{52}$$

Calculate the new (x) and (y) using the weighted average of (k)'s and (l)'s:

$$x_{n+1} = x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{53}$$

$$y_{n+1} = y_n + \frac{h}{6}(l_1 + 2l_2 + 2l_3 + l_4) \tag{54}$$

Continue steps 1-5 until reaching the desired endpoint. This is done by updating the (x) and (y) values at each iteration, and repeating the above steps until reaching the desired endpoint or the desired number of iterations.

2.3.3. Adams-Bashforth Method

To apply the Adams-Bashforth Method in the context of the Friedmann cosmological equation, we need to transform the differential equation into a form that can be numerically integrated. The Friedmann equation in cosmology describes the evolution of the universe and can be represented in general form as a differential equation that depends on parameters such as the Hubble rate of change {H(z)} with respect to redshift (z):

$$\frac{dH(z)}{dz} = f(z, H(z)) \tag{55}$$

In cosmology, the Hubble parameter (H(z)) shows how fast the universe is expanding at a time (z), while the function [f{z, H(z)}] determines how this expansion rate depends on time and the energy density in the universe. Using the Adams-Bashforth Method, it can estimate the value of the Hubble parameter at the next time step (z_{n+1}) by considering the previous values $\{H(z_n), H(z_{n-1}), \ldots\}$. This allows us to evaluate how the expansion of the universe evolves from one time step to the next by taking into account information from previous steps. This method helps in understanding the dynamics of the expanding universe in the context of changing time and energy density.

The numerical steps to solve the Friedmann equation start by choosing the number of time steps (N) and time steps (dz), which depend on the desired level of accuracy as well as the range of red dependence (z) to be traced. For example, if the goal is to integrate from (z=0) to $(z = z_{\max})$, the values of (N) and (dz) are chosen such that this range is divided into small time steps. Before applying the Adams-Bashforth Method, an initial value {H(z_0)} is required at (z_0). This value can be determined based on the initial conditions given in the cosmological problem or can be obtained from previous calculations. The next step is to calculate the value of {H(z_1)} using numerical methods such as the fourth-order Runge-Kutta method. This is done by solving the Friedmann differential equation at point (z_1), which will then be the initial value for the Adams-Bashforth Method iteration. After obtaining $\{H(z_0)\}$ and $\{H(z_1)\}$, the Adams-Bashforth equation is used to calculate the value of $\{H(z_{n+1})\}$ for each subsequent time step (n>1). This iterative equation takes into account the values of the function [f{z, H(z)] in the previous time steps to get an approximation of the value of $\{H(z_{n+1})\}$ by using a predetermined formula:

$$H(z_{n+1}) = H(z_n) + \frac{dz}{24}\begin{bmatrix} 55f\{z_n, H(z_n)\} - 59f\{z_{n-1}, H(z_{n-1})\} + 37f\{z_{n-2}, H(z_{n-2})\} \\ -9f\{z_{n-3}, H(z_{n-3})\} \end{bmatrix} \quad (56)$$

Where $\left[f\{z,\, H(z)\}\right]$ represents the function given in the Friedmann equation describing cosmological evolution. The iterative steps using the Adams-Bashforth Method are continued until reaching the last time step, (n = N), to obtain the numerical solution (H(z)) at the specified time steps (Zabidi et al., 2020). In a practical, after calculating the value of $\{H(z_{n+1})\}$ using the Adams-Bashforth Method, it is important to update the value of (z_{n+1}) and use the Friedmann equation to update $\{H(z_{n+1})\}$, because the calculated value of $\{H(z_{n+1})\}$ may not satisfy the Friedmann equation directly. This ensures the consistency of the numerical solution with the theoretical framework of cosmology given by the Friedmann equation.

3. Results

3.1. Influence of Cosmological Parameters in the Universe Expansion Rate Evolution Model

In this research, the first step is to substitute the value of (k) into the (Ω_k) equation to obtain the value of (Ω_k). The substitution results in a new equation for (Ω_k), namely $\left(\Omega_k = 16\pi^2 G^2 \rho_0^2 / 3\Lambda a_0^2 H_0^2\right)$. In terms of physics, it describes the relationship between cosmological parameters such as the universal gravitational constant (G), the average density of the universe (ρ_0), the cosmological constant (Λ), as well as other parameters such as the current scale scalar a_0 and the current expansion rate (H_0). The determined parameters, namely (Ω_k), (Ω_Λ), and (Ω_m), will be used to replace the variables in the cosmological equation. The equation for calculating the expansion rate of the universe, {H(z)}, relative to the current expansion rate, (H_0), is given by:

$$\frac{H^2(z)}{H_0^2} = \frac{8\pi G\rho_0}{3H_0^2}(1+z)^3 + \frac{16\pi^2 G^2 \rho_0^2}{3\Lambda a_0^2 H_0^2}(1+z)^2 + \frac{\Lambda}{3H_0^2}. \quad (57)$$

Where (z) is the redshift, (ρ_0) is the current average matter density, (G) is the gravitational constant, and (Λ) is the cosmological constant.

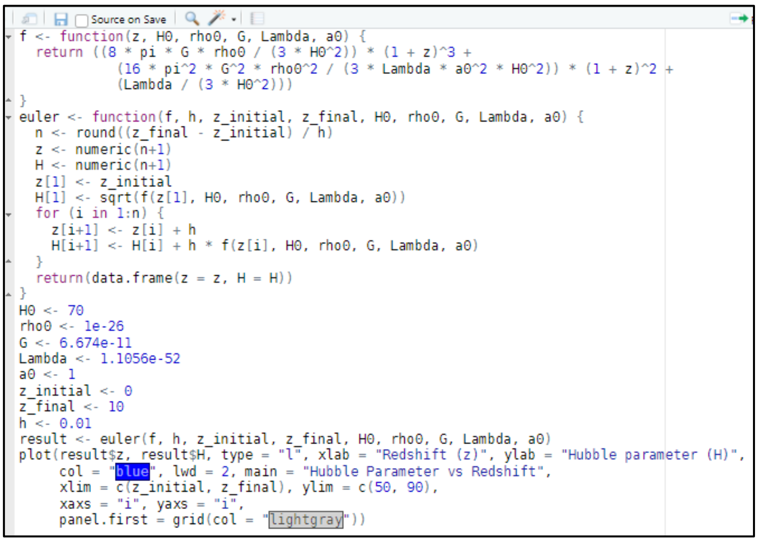
3.2. Test the Universe Expansion Rate Evolution Model using numerical methods

3.2.1. Euler numerical integration test

In the code used, which can be seen in the figure below, we start by initializing physics parameters such as the Hubble constant $(H_0 = 70\text{km/s/Mpc})$, density parameter $(\rho_0 = 1\times 10^{-26}\,\text{kg/m}^3)$, gravitational constant $(G = 6.674\times 10^{-11}\,\text{m}^3/\text{kg/s}^2)$, the cosmological constant $\{f(z,\, H_0,\, \rho_0,\, G,\, \Lambda,\, a_0)\}$, as well as today's scale factor (a_0 = 1).

Then, the algorithm starts by defining a function $\{f(z, H_0, \rho_0, G, \Lambda, a_0)\}$ that represents the given differential equation in the context of cosmological physics. This function uses the variables (z) as the redshift, (H_0) as the Hubble constant, ρ_0 as the density parameter, (G) as the gravitational constant, (Λ) as the cosmological constant, and a_0 as the scale factor of the day. Next, an euler function is defined to implement the Euler method of solving differential equations. This function takes arguments such as steps (h), initial value (z_{initial}), and final value (z_{final}), as well as other physics parameters.

The iterative steps are calculated based on the initial and final values of (z) and the (h) steps. The variables (z) and (H) are initialized with appropriate numeric arrays. The iterative process is performed using a for loop for (n) times, by calculating the values of (z) and (H) using the iterative formula of the Euler method. Each iteration, the value of (z) is updated by the (h) step, and the value of (H) is updated using the Euler method approximation. The numerical results of the differential equations are stored in a data frame containing the values of (z) and (H) at each step. The numerical results are plotted using the plot function of R to visualize the relationship between the Hubble parameter and redshift.



```r
f <- function(z, H0, rho0, G, Lambda, a0) {
  return ((8 * pi * G * rho0 / (3 * H0^2)) * (1 + z)^3 +
            (16 * pi^2 * G^2 * rho0^2 / (3 * Lambda * a0^2 * H0^2)) * (1 + z)^2 +
            (Lambda / (3 * H0^2)))
}
euler <- function(f, h, z_initial, z_final, H0, rho0, G, Lambda, a0) {
  n <- round((z_final - z_initial) / h)
  z <- numeric(n+1)
  H <- numeric(n+1)
  z[1] <- z_initial
  H[1] <- sqrt(f(z[1], H0, rho0, G, Lambda, a0))
  for (i in 1:n) {
    z[i+1] <- z[i] + h
    H[i+1] <- H[i] + h * f(z[i], H0, rho0, G, Lambda, a0)
  }
  return(data.frame(z = z, H = H))
}
H0 <- 70
rho0 <- 1e-26
G <- 6.674e-11
Lambda <- 1.1056e-52
a0 <- 1
z_initial <- 0
z_final <- 10
h <- 0.01
result <- euler(f, h, z_initial, z_final, H0, rho0, G, Lambda, a0)
plot(result$z, result$H, type = "l", xlab = "Redshift (z)", ylab = "Hubble parameter (H)",
     col = "blue", lwd = 2, main = "Hubble Parameter vs Redshift",
     xlim = c(z_initial, z_final), ylim = c(50, 90),
     xaxs = "i", yaxs = "i",
     panel.first = grid(col = "lightgray"))
```

Figure 1. Simulation of Euler Method for Calculation of Hubble vs Redshift Parameters in Cosmology

The results illustrates the correlation between redshift (z) and the Hubble parameter (H), serving as an indicator of the universe's expansion rate in cosmology. Redshift (z) quantifies the distance of celestial objects from the observer, with higher values indicating greater distances. Remarkably, the Hubble parameter (H) remains consistent across the entire redshift (z) spectrum in the result. This constancy suggests an unaltered universe expansion within the implemented cosmological framework, devoid of external influences like gravitational forces or dark energy. Consequently, the results confirms that, within the specified model parameters, the universe's expansion remains uniform over the observed period.

3.2.2. Runge-kutta numerical test

In the realm of cosmological physics, the function `f` encapsulates the essence of cosmic evolution. Taking into account the redshift (`z`), the Hubble parameter (`H`), and a set of crucial physical parameters stored in an array `params`, this function computes the differential equation's value, symbolizing the dynamic progression of the universe. The outcome of `f` yields the square root of the right-hand side of the equation, offering insights into how the Hubble parameter evolves with shifting redshifts, thereby unraveling the universe's rate of expansion.

```
Source on Save
f <- function(x, y, params) {
  H0 <- params$H0
  G <- params$G
  rho0 <- params$rho0
  Lambda <- params$Lambda
  a0 <- params$a0
  f <- c(
    y[2],
    (8 * pi * G * rho0 / (3 * H0^2)) * (1 + x)^3 + (16 * pi^2 * G^2 * rho0^2 / (3 * Lambda * a0^2 * H0^2))
    * (1 + x)^2 + (Lambda / (3 * H0^2))
  )
  return(f)
}
runge_kutta <- function(f, x0, y0, params, h, n) {
  x <- x0
  y <- matrix(0, nrow = n + 1, ncol = length(y0))
  y[1,] <- y0
  for (i in 1:n) {
    k1 <- h * f(x, y[i,], params)
    k2 <- h * f(x + h/2, y[i,] + k1/2, params)
    k3 <- h * f(x + h/2, y[i,] + k2/2, params)
    k4 <- h * f(x + h, y[i,] + k3, params)
    y[i + 1,] <- y[i,] + (k1 + 2*k2 + 2*k3 + k4) / 6
    x <- x + h
  }
  return(data.frame(x = seq(x0, x0 + n * h, by = h), y = y))
}
params <- list(
  H0 = 70,
  G = 6.674e-11,
  rho0 = 1e-26,
  Lambda = 1e-52,
  a0 = 1
)
x0 <- 0
y0 <- c(1, 0)
h <- 0.01
n <- 100 #
result <- runge_kutta(f, x0, y0, params, h, n)
plot(result$x, result$y[,1], type="l", xlab="z", ylab="H^2(z)/H0^2", main="Numerical Solution using Runge-Kutta")
```

Figure 2. Algorithm of R program Simulation of runge-kutta Method for model

The `runge_kutta` function described above facilitates the numerical solution of a given physical system by employing the 4th-order Runge-Kutta method. This function takes as input the differential equation function `f`, initial values of the redshift `z0` and the Hubble parameter `$H_0$`, along with fundamental physical parameters such as gravitational and cosmological constants. It also requires specifications for the step size `h` and the number of steps `n` utilized in the numerical approximation. During each iteration, the method computes intermediate values `$k_1$`, `$k_2$`, `$k_3$`, and `$k_4$` based on the provided differential equation function `f` evaluated at predetermined points. Subsequently, it updates the values of the redshift `z` and the Hubble parameter `H` accordingly. The resulting output is a data frame containing the redshift `z` and the corresponding Hubble parameter `H` at each simulation step, enabling further analysis of how these physical parameters evolve as the redshift changes.

The simulation results illustrate how the Hubble parameter (H) evolves with respect to changes in redshift (z), which is an indicator of the spectral shift of light produced by changes in the wavelength of observed objects in the universe. From the result, it can be seen that the higher the redshift, the faster the Hubble parameter changes, reflecting the dynamics of the expansion of the Universe.

From the result, we can see that as redshift increases, the Hubble parameter also increases, in accordance with the cosmological model that describes the expansion of the Universe. The higher the redshift, the faster the Universe is expanding. The physics parameters used in the simulation, such as the Hubble constant (H_0), gravitational constant (G), density parameter (ρ_0), and cosmological constant (Λ), significantly affect the rate of change of the Hubble parameters as the redshift changes.

3.2.3. Adams-Bashforth numerical test

The function `f(z, H, rho0, G, Lambda, H0)` is used to calculate the derivative of H(z) in the context of a differential equation modeling the change in the expansion rate of the universe at the point (z, H). The function `adams_bashforth(z0, H0, rho0, G, Lambda, h, num_steps)` is used to iterate with the Adams-Bashforth method, where the steps integrate the derivative to estimate the value of H(z) at subsequent points.

Determine initial values for physical parameters such as the cosmological redshift (z_0) and the expansion rate of the universe ($H(z_0)$). Iterate a predetermined number of steps to estimate the value of H(z) at subsequent points. At each iteration, Use the Runge-Kutta method of order 4 (RK4) to calculate the first step, which helps us in estimating the value of H(z) at the next point. After that, apply the Adams-Bashforth method to iterate the value of H(z) at subsequent points using the previous values. During the iteration process, we also update the cosmological redshift value (z) to describe the evolution of the universe over time. After obtaining the numerical results, we plot the result of H(z) against z using the `plot()` function.

```
plot(result$x, result$y[,1], type="l", xlab="z", ylab="H^2(z)/H0^2", main="Numerical Solution using Runge-Kutta")

f <- function(z, H, rho0, G, Lambda, H0) {
  term1 <- (8 * pi * G * rho0 / (3 * H0^2)) * (1 + z)^3
  term2 <- (16 * pi^2 * G^2 * rho0^2 / (3 * Lambda * H0^2)) * (1 + z)^2
  term3 <- Lambda / (3 * H0^2)
  return(sqrt(term1 + term2 + term3))
}
adams_bashforth <- function(z0, H0, rho0, G, Lambda, h, num_steps) {
  z <- numeric(num_steps)
  H <- numeric(num_steps)
  z[1] <- z0
  H[1] <- H0
  for (i in 2:num_steps) {
    k1 <- h * f(z[i-1], H[i-1], rho0, G, Lambda, H0)
    k2 <- h * f(z[i-1] + h/2, H[i-1] + k1/2, rho0, G, Lambda, H0)
    k3 <- h * f(z[i-1] + h/2, H[i-1] + k2/2, rho0, G, Lambda, H0)
    k4 <- h * f(z[i-1] + h, H[i-1] + k3, rho0, G, Lambda, H0)
    H_pred <- H[i-1] + (k1 + 2*k2 + 2*k3 + k4) / 6
    if (i == 2) {
      H[i] <- H_pred
    } else {
      H[i] <- H[i-1] + (h / 2) * (3 * f(z[i-1], H[i-1], rho0, G, Lambda, H0) - f(z[i-2], H[i-2], rho0, G, Lambda, H0))
    }
    z[i] <- z[i-1] + h
  }
  return(data.frame(z = z, H = H))
}
z0 <- 0
H0 <- 70
rho0 <- 1e-26
G <- 6.674e-11
Lambda <- 1e-52
h <- 0.01
num_steps <- 1000
result <- adams_bashforth(z0, H0, rho0, G, Lambda, h, num_steps)
plot(result$z, result$H, type = "l", xlab = "z", ylab = "H(z)", main = "Solusi Numerik dengan Metode Adams-Bashforth")
```

Figure 3. Algorithm of R program Simulation of Adams-Bashforth Method for model

4. Conclussion

This research explores the importance of understanding cosmological parameters in evolutionary models of the expansion rate of the universe. We test these models using numerical methods such as Euler, Runge-Kutta, and Adams-Bashforth. Through Euler testing, we observe how the initialization of physics parameters such as the Hubble constant H_0, density parameter (ρ), gravitational constant (G), and cosmological constant Λ, affect the iterative calculation of the expansion rate of the universe. Our numerical results provide a clear visualization of the relationship between the Hubble parameter and redshift, which is important in understanding the change in expansion rate. Testing with the Runge-Kutta method provides deep insight into the evolution of physical parameters as redshift changes. Using the (f) function that reflects cosmic evolution, we estimate the values of (z) and (H) at each simulation step. The visualization results show how the physics parameters behave as redshift changes, providing a better understanding of the expansion dynamics of the Universe. The final test with the Adams-Bashforth method makes an additional contribution to understanding the evolution of the expansion rate of the Universe. With this method, we can estimate the value of (H(z)) at later points more accurately. Visualization of the numerical results of (H(z)) against (z) shows how the expansion rate of the universe evolves along with the change of redshift. This study confirms that models of the expansion rate evolution of the universe produce a consistent relationship between cosmological parameters and the expansion rate, as observed through the various numerical methods used. This makes a significant contribution to our understanding of the dynamics of the Universe and the evolution of its parameters through time.

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