Bound state solution of schrodinger equation with modified Hylleraas plus inversely quadratic potential using parametric Nikiforov-Uvarov method

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Received 21 November 2023, Revised 15 March 2024, Published 30 March 2024

Abstract: In this work, an approximate bound state solution to Schrodinger equation with modified Hylleraas plus inversely quadratic potential was obtained using the parametric form of the Nikiforov-Uvarov method. Applying the elegant approximation to deal with the centrifugal term for arbitrary orbital angular quantum number, the energy spectrum and the corresponding wave function was obtained.

Keyword : Nikiforov-Uvarov Method, centrifugal term, Modified Hylleraass plus Inversely Quadratic potential.

1. Introduction

Schrodinger wave equation belongs to non-relativistic wave equation. The total wave function of any quantum mechanical system basically provides implicitly the relevant information about the physical behavior of the system (Okon et al., 2013; Okon et al., 2014; Okon et al., 2014; Okon et al., 2015).The bound state solutions of Schrodinger equation in quantum mechanics is very difficult to solve with some physical central potential (Ikot et al., 2011; Hassanabadi et al., 2011). Recently, the study of exponential-type potential has attracted a lot of interest by many authors (Ikot & Akpabio, 2010; Ikhdair, 2011). Some of the potentials under consideration are: woods-Saxon plus modified exponential coulomb potential, Hulthen plus generalized exponential coulomb potential, Rosen-Morse, Hulthen, pseudo harmonic, Posch-Teller, kratzerfues and Mie-Type potential and Eckart potential etc. Different analytical techniques have been adopted by different authors in providing solutions to relativistic and non- relativistic wave equations. Moreover, with the arbitrary angular momentum quantum number, one can only solve the Schrodinger equation approximately using a suitable approximation scheme (Ciftci et al., 2003; Ikot & Akpabio, 2010; Yahya & Oyewumi, 2016; Edet et al., 2019). One of such approximation includes conventional approximation scheme proposed by Greene and Aldrich (2012), improved approximation scheme by Qiang et al. (2007), elegant approximation scheme (Ikot et al., 2016) and a new approximation scheme by Dong et al. (2007). These approximations are used to deal with the centrifugal term and many authors have investigated approximately the bound state solutions of the Schrodinger equation with exponential-like potentials. For further details readers can refer to most recent works (Antia et al., 2010; Sun et al., 2013; Ikot et al., 2013). In this work, elegant approximation scheme is used to deal with the centrifugal term and solve Schrodinger equation with modified Hylleraas plus the modified Hylleraas plus inversely quadratic potential defined as (Ituen et al., 2015).

$$V(r) = -\frac{v_0}{\delta_2}\left(\frac{\delta_1 e^{-2\alpha r}}{1-e^{-2\alpha r}}\right) - \frac{\delta_3}{r^2} \tag{1}$$

where, v_0 and δ_3 are constant called potential depth, δ_1 and δ_2 are Hylleraas parameter, α is the adjustable parameter and r is the molecular distance.

2. Review of Nikiforov-Uvarov (NU) Method and its Parametric Form

The NU method is based on solving a second-order linear differential equation by reducing it to a generalized equation of hypergeometric-type. The NU method has been used to solve the Schrödinger, Dirac and Klein-Gordon wave equation for a certain kind of potential (Nikiforov & Uvarov, 1988; Ikot et al., 2011; Ikot et al., 2011). The NU equation is given as

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \tag{2}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree, and $\tilde{\tau}(s)$ is a first degree polynomial. In order to find a particular solution to Equation (2), we use the following transformation.

$$\psi(s) = \varphi(s)\chi(s) \tag{3}$$

This reduces Equation (2) to an equation of hyper geometric type,

$$\sigma(s)\chi''(s) + \tau(s)\chi'(s) + \lambda\chi(s) = 0 \tag{4}$$

where $\varphi(s)$ is defined as a logarithmic derivative

$$\frac{\varphi'(s)}{\varphi(s)} = \frac{\pi(s)}{\sigma(s)} \tag{5}$$

The other part $\chi(s)$ is the hyper geometric type function whose polynomials are given by the Rodrigues relation,

$$\chi_n(s) = \frac{B_n(s)}{\alpha(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right] \tag{6}$$

where B_n is a normalization constant and the weight function $\alpha(s)$ must satisfy the condition

$$\frac{d}{ds}\left(\sigma(s)\rho(s)\right) = \tau(s)\rho(s) \tag{7}$$

With

$$\tau(s)=\tilde{\tau}(s)+2\pi(s) \tag{8}$$

In order to accomplish the condition imposed on the weight function $\rho(s)$, it is necessary that the classical or polynomials $\tau(s)$ be equal to zero to some point of an interval (a, b) and its derivative at this interval at $\sigma'(s)>0$ will be negative, that is

$$\frac{d\sigma(s)}{ds}<0 \tag{9}$$

The function $\pi(s)$ and the parameter λ required for the NU method are defined as follows:

$$\pi(s)=\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\pm\sqrt{\left(\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\right)^2-\tilde{\sigma}(s)+k\sigma(s)} \tag{10}$$

$$\lambda=k+\pi'(s) \tag{11}$$

On the other hand, in order to find the value of k, in equation (10) the expression under the square root must be the square of the polynomial. This is possible, if and only if its discriminant is zero. Thus, a new eigenvalue for the second-order differential equation becomes

$$\lambda=\lambda_n=-n\tau'(s)-\frac{n(n-1)}{2}\sigma''(s) \quad \text{for } n=0,1,2\,..... \tag{12}$$

By the comparison Equation (11) and Equation (12), we obtain the energy eigenvalues.

The parametric generalization of the NU method is given by the generalized hypergeometric-type equation given as (Hill & Phys, 1954; Ikot et al, 2013; Akpan et al., 2018)

$$\psi''(s)+\frac{(c_1-c_2s)}{s(1-c_3s)}\psi'(s)+\frac{1}{s^2(1-c_3s)^2}\left\{-\xi_1s^2+\xi_1s-\xi_1\right\}\psi(s)=0 \tag{13}$$

Solving Equation (13) is by comparing it with Equation (2) and the following polynomials are obtained:

$$\left.\begin{aligned}\tilde{\tau}(s)&=(c_1-c_2s)\\ \sigma(s)&=s(c_1-c_3s)\\ \tilde{\sigma}(s)&=-\xi_1s^2+\xi_1s-\xi_1\end{aligned}\right\} \tag{14}$$

Differentiate $\tau(s)$ and $\sigma(s)$ gives

$$\left.\begin{aligned}\tilde{\tau}'(s)&=-c_2\\ \sigma'(s)&=c_1-2c_3s\end{aligned}\right\} \tag{15}$$

Substitute equation (14) and (15) into equation (10) we have

$$\pi(s)=\frac{1-c_1-2c_3s+c_2}{2}\pm\sqrt{\left(\frac{1-c_1-2c_3s+c_2}{2}\right)^2-\left(-\xi_1s^2+\xi_1s-\xi_1\right)+k\left(s(c_1-c_3s)\right)} \tag{16}$$

$$\pi(s)=\frac{1-c_1-2c_2s+c_2}{2}\pm\sqrt{\left(\frac{c_1-2c_2s+c_2}{2}\right)^2-\left(-\xi_1s^2+\xi_1s-\xi_1\right)+k\left(s\left(c_1-c_2s\right)\right)} \tag{17}$$

$$\pi(s)=c_4+c_5s\pm\sqrt{\left(c_6-c_3k_\pm\right)s^2+\left(c_7+k_\pm\right)s+c_8} \tag{18}$$

Or

$$\pi(s)=c_4+c_5s\pm\left[\left(c_6-c_3k_\pm\right)s^2+\left(c_7+k_\pm\right)s+c_8\right]^{\frac{1}{2}} \tag{19}$$

where

$$c_4=\frac{1}{2}\left(1-c_1\right), c_5=\frac{1}{2}\left(c_2-2c_3\right), c_6=c_5^2+\xi_1, c_7=2c_4c_5-\xi_2, c_8=c_4^2+\xi_3 \tag{20}$$

The resulting value of k in Equation (19) is obtained from the condition that the function under the square root be square of a polynomials and it yields

$$k_\pm=-\left(c_7+2c_3c_8\right)\pm2\sqrt{c_8c_9} \tag{21}$$

where

$$c_9=c_3c_7+c_2^2c_8+c_6 \tag{22}$$

The new $\pi(s)$ for k_- becomes

$$\pi(s)=c_4+c_5s-\left\{\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)s-\sqrt{c_8}\right\} \tag{23}$$

for k_- value

$$k-=-\left(c_7+2c_3c_8\right)-2\sqrt{c_8c_9} \tag{24}$$

Using equation (14) and (23) into equation (8) we have

$$\tau(s)=\left(c_1-c_2s\right)+2\left(c_4+c_5s-\left\{\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)s-\sqrt{c_8}\right\}\right) \tag{25}$$

$$\tau(s)=c_1+2c_4-\left(c_2-c_5\right)s-2\left\{\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)s-\sqrt{c_8}\right\} \tag{26}$$

But the physical condition for the bound state solution is $\tau<0$ and thus

$$\tau'(s)=-2c_3-2\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)<0 \tag{27}$$

With the aid of Equation (11) and (12), we obtain the energy equation as

$$\left(c_2-c_3\right)n+c_3n^2-\left(2n+1\right)c_5+\left(2n+1\right)\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)+c_7+2c_3c_8+2\sqrt{c_8c_9}=0 \tag{28}$$

Using equation (7) the weight function $\rho(s)$ is obtained as

$$\rho(s)=s^{c_{10}-1}\left(1-c_3s\right)^{\frac{c_{11}}{c_3}-c_{10}-1} \tag{29}$$

Substitute equation (10) into (6) gives

$$\chi(s)=P_n^{\left(c_{10}-1,\frac{c_{11}}{c_3}-c_{10}-1\right)}\left(1-2c_3s\right) \tag{30}$$

$P_n^{(\in,\vartheta)}(s)$ are the Jacobi Polynomials.

where

$$c_{10}=c_1+2c_4+2\sqrt{c_8},\quad c_{11}=c_2-2c_5+2\left(\sqrt{c_9}+c_3\sqrt{c_8}\right) \tag{31}$$

The second part of the wave function is obtained from Equation (5) as

$$\varphi(s) = s^{c_{12}}(1-c_3 s)^{-c_{12}-\frac{c_{13}}{c_3}} \tag{32}$$

where

$$c_{12} = c_4 + \sqrt{c_8}, \quad c_{13} = c_5 - 2\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) \tag{33}$$

Thus the total wave function for parametric NU method is obtain using equation (3)

$$\psi(s) = N_n s^{c_{12}}(1-c_3 s)^{-c_{12}-\frac{c_{13}}{c_3}} \times P_n^{\left(c_{10}-1,\ \frac{c_{11}}{c_3}-c_{10}-1\right)}(1-2c_3 s) \tag{34}$$

where N_n is the normalization constant.

3. Radial Solutions of Schrödinger Equation

The radial part of Schrodinger equation with the potential $V(r)$ is given as (Ikot et al., 2011).

$$\frac{d^2R_{nl}(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[E_{nl} - V(r) - \frac{l(l+1)\hbar^2}{2\mu r^2}\right]R_{nl}(r) = 0 \tag{35}$$

Substituting potential of Equation (1) into equation (35), we obtain

$$\frac{d^2R_{nl}(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[E_{nl} + \frac{v_0}{\delta_2}\left(\frac{\delta_1 e^{-2\alpha r}}{1-e^{-2\alpha r}}\right) + \frac{\delta_3}{r^2} - \frac{l(l+1)\hbar^2}{2\mu r^2}\right]R_{nl}(r) = 0 \tag{36}$$

Equation (36) cannot be solved exactly for $l \neq 0$ by any known method due to centrifugal term $1/r^2$. Let invoke the elegant approximation (Ikot et al., 2011) to deal with the centrifugal term $1/r^2$.

$$\frac{1}{r^2} \approx \alpha^2\left[D_0 + \frac{D_1}{(1-e^{-2\alpha r})} + \frac{D_2}{(1-e^{-2\alpha r})^2}\right] \tag{37}$$

Substituting equation (37) into (36) gives

$$\frac{d^2R_{nl}(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[E_{nl} + \frac{v_0}{\delta_2}\left(\frac{\delta_1 e^{-2\alpha r}}{1-e^{-2\alpha r}}\right) + \delta_3\left(\alpha^2\left[D_0 + \frac{D_1}{(1-e^{-2\alpha r})} + \frac{D_2}{(1-e^{-2\alpha r})^2}\right]\right)R_{nl}(r)\right]$$
$$-\frac{2\mu}{\hbar^2}\left[\frac{l(l+1)\hbar^2}{2\mu}\left(\alpha^2\left[D_0 + \frac{D_1}{(1-e^{-2\alpha r})} + \frac{D_2}{(1-e^{-2\alpha r})^2}\right]\right)R_{nl}(r)\right] = 0 \tag{38}$$

$$\frac{d^2R_{nl}(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[E_{nl} + \frac{v_0\delta_1 e^{-2\alpha r}}{\delta_2(1-e^{-2\alpha r})} + \delta_3\alpha^2 D_0 + \frac{D_1\delta_3\alpha}{(1-e^{-2\alpha r})} + \frac{\delta_3\alpha^2 D_2}{(1-e^{-2\alpha r})^2}\right]R_{nl}(r)$$
$$-\frac{2\mu}{\hbar^2}\left[\alpha^2 D_0\frac{l(l+1)\hbar^2}{2\mu} + \frac{\alpha^2 D_1}{(1-e^{-2\alpha r})}\frac{l(l+1)\hbar^2}{2\mu} + \frac{\alpha^2 D_2}{(1-e^{-2\alpha r})^2}\frac{l(l+1)\hbar^2}{2\mu}\right]R_{nl}(r) = 0 \tag{39}$$

With coordinate transformation

$$s = -e^{-2\alpha r} \tag{40}$$

Equation (39) gives

$$\frac{d^2R_{nl}(s)}{ds^2}+\frac{1}{s}\frac{dR_{nl}(s)}{ds}+\frac{1}{4\alpha^2s^2}\frac{2\mu}{\hbar^2}\left[E_{nl}-\frac{v_0\delta_1 s}{\delta_2(1-s)}+\delta_3\alpha^2D_0+\frac{D_1\delta_3\alpha}{(1-s)}+\frac{\delta_3\alpha^2D_2}{(1-s)^2}\right]R_{nl}(s)$$
$$-\frac{1}{4\alpha^2s^2}\frac{2\mu}{\hbar^2}\left[\alpha^2D_0\frac{l(l+1)\hbar^2}{2\mu}+\frac{\alpha^2D_1}{(1-s)}\frac{l(l+1)\hbar^2}{2\mu}+\frac{\alpha^2D_2}{(1-s)^2}\frac{l(l+1)\hbar^2}{2\mu}\right]R_{nl}(s)=0 \tag{41}$$

$$\frac{d^2R_{nl}(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{dR_{nl}(s)}{ds}$$
$$+\frac{1}{s^2(1-s)^2}\begin{bmatrix}\frac{\mu E_{nl}}{2\alpha^2\hbar^2}-\frac{\mu E_{nl}}{\alpha^2\hbar^2}s+\frac{\mu E_{nl}}{2\alpha^2\hbar^2}s^2-\frac{\mu v_0\delta_1}{2\alpha^2\hbar^2\delta_2}s+\frac{\mu v_0\delta_1}{2\alpha^2\hbar^2\delta_2}s^2\\ +\frac{\mu\delta_3D_0}{2\hbar^2}-\frac{\mu\delta_3D_0}{\hbar^2}s+\frac{\mu\delta_3D_0}{2\hbar^2}s^2+\frac{\mu D_1\delta_3}{2\alpha\hbar^2}-\frac{\mu D_1\delta_3}{2\alpha\hbar^2}s-\frac{\mu\delta_3D_2}{2\hbar^2}\\ -\frac{D_0l(l+1)}{4}+\frac{D_0l(l+1)}{2}s-\frac{D_0l(l+1)}{4}s^2-\frac{D_1l(l+1)}{4}\\ +\frac{D_1l(l+1)}{4}s-\frac{D_2l(l+1)}{4}\end{bmatrix}R_{nl}(s)=0 \tag{42}$$

By simplifying and rearrange equation (42) gives

$$\frac{d^2R_{nl}(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{dR_{nl}(s)}{ds}$$
$$+\frac{1}{s^2(1-s)^2}\begin{bmatrix}\frac{\mu E_{nl}}{2\alpha^2\hbar^2}+\left(\frac{\mu E_{nl}}{2\alpha^2\hbar^2}+\frac{\mu v_0\delta_1}{2\alpha^2\hbar^2\delta_2}+\frac{\mu\delta_3D_0}{2\hbar^2}-\frac{D_0l(l+1)}{4}\right)s^2\\ -\left(\frac{\mu E_{nl}}{\alpha^2\hbar^2}+\frac{\mu v_0\delta_1}{2\alpha^2\hbar^2\delta_2}+\frac{\mu\delta_3D_0}{\hbar^2}+\frac{\mu D_1\delta_3}{2\alpha\hbar^2}-\frac{D_0l(l+1)}{2}-\frac{D_1l(l+1)}{4}\right)s\\ +\frac{\mu\delta_3D_0}{2\hbar^2}+\frac{\mu D_1\delta_3}{2\alpha\hbar^2}-\frac{\mu\delta_3D_2}{2\hbar^2}-\frac{D_0l(l+1)}{4}-\frac{D_1l(l+1)}{4}-\frac{D_2l(l+1)}{4}\end{bmatrix}R_{nl}(s)=0 \tag{43}$$

Let

$$-\varepsilon^2=\frac{\mu E_{nl}}{2\alpha^2\hbar^2} \tag{44}$$

$$A=\frac{\mu}{2\alpha^2\hbar^2\delta_2}\left(v_0\delta_1+\delta_2\delta_3\alpha^2D_0\right)-\frac{D_0l(l+1)}{4} \tag{45}$$

$$B=-\frac{\mu v_0\delta_1}{2\alpha^2\hbar^2\delta_2}-\frac{\mu\delta_3}{2\hbar^2}(2D_0+D_1)-\frac{l(l+1)}{2}\left(D_0+\frac{D_1}{2}\right) \tag{46}$$

$$C=\frac{\mu\delta_3}{2\alpha\hbar^2}(\alpha D_0+D_1-\alpha D_2)-\frac{l(l+1)}{4}(D_0+D_1+D_2) \tag{47}$$

Using equation (45), (56) and (47) into (43) yield

$$\frac{d^2R_{nl}(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{dR_{nl}(s)}{ds}+\frac{1}{s^2(1-s)^2}\left[-\left(\varepsilon^2-A\right)s^2+\left(2\varepsilon^2+B\right)s-\left(\varepsilon^2-C\right)\right]R_{nl}(s)=0 \tag{48}$$

Equation (48) is similar to the parametric generalization of NU method given from equation (13) above.

By compare equation (48) with equation (13) we have the following parameter

$$\xi_1=\varepsilon^2-A,\ \xi_2=2\varepsilon^2+B,\ \xi_3=\varepsilon^2-C \tag{49}$$

$$\left.\begin{aligned}
&c_1=c_2=c_3=1\\
&c_4=\frac{1}{2}(1-c_1)=0\\
&c_5=\frac{1}{2}(c_2-2c_3)=\frac{1}{2}(1-2)=-\frac{1}{2}\\
&c_6=c_5^2+\xi_1=\left(-\frac{1}{2}\right)^2+\varepsilon^2-A=\frac{1}{4}+\varepsilon^2-A\\
&c_7=2c_4c_5-\xi_2=-\xi_2=-\left(2\varepsilon^2-B\right)\\
&c_8=c_4^2+\xi_3=\xi_3=\varepsilon^2-C\\
&c_9=c_3c_7+c_2^2c_8+c_6=-2\varepsilon^2-B+\varepsilon^2-C+\frac{1}{4}+\varepsilon^2-A=\frac{1}{4}-(A+B+C)\\
&c_{10}=c_1+2c_4+2\sqrt{c_8},=1+2\sqrt{\varepsilon^2-C}\\
&c_{11}=c_2-2c_5+2\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)=2+2\left\{\sqrt{\frac{1}{4}-A-B-C}+\sqrt{\varepsilon^2-C}\right\}\\
&c_{12}=c_4+\sqrt{c_8},=\sqrt{\varepsilon^2-C}\\
&c_{13}=c_5-2\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)=-\frac{1}{2}-\left(\sqrt{\frac{1}{4}-A-B-C}+\sqrt{\varepsilon^2-C}\right)
\end{aligned}\right\} \tag{50}$$

Substitute the value of equation (50) into (28) to obtain the energy spectrum of the system

$$(1-1)n+n^2-(2n+1)\left(-\frac{1}{2}\right)+(2n+1)\left(\sqrt{\frac{1}{4}-A-B-C}+\sqrt{\varepsilon^2-C}\right)-\left(2\varepsilon^2-B\right)+2\left(\varepsilon^2-C\right)$$
$$+2\sqrt{\left(\varepsilon^2-C\right)\left(\frac{1}{4}-A-B-C\right)}=0 \tag{51}$$

$$\varepsilon^2=C+\left[\frac{-(B-2C)-\left[\left(n+\frac{1}{2}\right)+\sqrt{\frac{1}{4}-A-B-C}\right]^2}{\left\{n+\frac{1}{2}+\sqrt{\left(\frac{1}{4}-A-B-C\right)}\right\}}\right]^2 \tag{52}$$

J. Phys.: Theor. Appl. Vol. 8 No. 1 (2024) 25-36 doi: 10.20961/jphystheor-appl.v8i1.80582

Using equation (44), (45), (46) and (47) into (52) gives

$$-\frac{\mu E_{nl}}{2\alpha^2\hbar^2}=\frac{\mu\delta_3}{2\alpha\hbar^2}\left(\alpha D_0+D_1-\alpha D_2\right)-\frac{l(l+1)}{4}\left(D_0+D_1+D_2\right)$$

$$+\left[\frac{\begin{array}{l}-\left(-\frac{\mu v_0\delta_1}{2\alpha^2\hbar^2\delta_2}-\frac{\mu\delta_3}{2\hbar^2}\left(2D_0+D_1\right)-\frac{l(l+1)}{2}\left(D_0+\frac{D_1}{2}\right)\right)\\ -2\left(\frac{\mu\delta_3}{2\alpha\hbar^2}\left(\alpha D_0+D_1-\alpha D_2\right)-\frac{l(l+1)}{4}\left(D_0+D_1+D_2\right)\right)\\ -\left[\left(n+\frac{1}{2}\right)+\sqrt{\begin{array}{l}\frac{1}{4}-\left(\frac{\mu}{2\alpha^2\hbar^2\delta_2}\left(v_0\delta_1+\delta_2\delta_3\alpha^2 D_0\right)-\frac{D_0 l(l+1)}{4}\right)\\ -\left(-\frac{\mu v_0\delta_1}{2\alpha^2\hbar^2\delta_2}-\frac{\mu\delta_3}{2\hbar^2}\left(2D_0+D_1\right)-\frac{l(l+1)}{2}\left(D_0+\frac{D_1}{2}\right)\right)\\ -\left(\frac{\mu\delta_3}{2\alpha\hbar^2}\left(\alpha D_0+D_1-\alpha D_2\right)-\frac{l(l+1)}{4}\left(D_0+D_1+D_2\right)\right)\end{array}}\right]^2\end{array}}{\left\{\left(n+\frac{1}{2}\right)+\sqrt{\begin{array}{l}\frac{1}{4}-\left(\frac{\mu}{2\alpha^2\hbar^2\delta_2}\left(v_0\delta_1+\delta_2\delta_3\alpha^2 D_0\right)-\frac{D_0 l(l+1)}{4}\right)\\ -\left(-\frac{\mu v_0\delta_1}{2\alpha^2\hbar^2\delta_2}-\frac{\mu\delta_3}{2\hbar^2}\left(2D_0+D_1\right)-\frac{l(l+1)}{2}\left(D_0+\frac{D_1}{2}\right)\right)\\ -\left(\frac{\mu\delta_3}{2\alpha\hbar^2}\left(\alpha D_0+D_1-\alpha D_2\right)-\frac{l(l+1)}{4}\left(D_0+D_1+D_2\right)\right)\end{array}}\right\}}\right]^2 \tag{53}$$

Let make

$$\eta=\frac{\mu\delta_3}{2\alpha\hbar^2}\left(\alpha D_0+D_1-\alpha D_2\right)-\frac{l(l+1)}{4}\left(D_0+D_1+D_2\right) \tag{54}$$

$$\gamma=-\left(-\frac{\mu v_0\delta_1}{2\alpha^2\hbar^2\delta_2}-\frac{\mu\delta_3}{2\hbar^2}\left(2D_0+D_1\right)-\frac{l(l+1)}{2}\left(D_0+\frac{D_1}{2}\right)\right)$$
$$-2\left(\frac{\mu\delta_3}{2\alpha\hbar^2}\left(\alpha D_0+D_1-\alpha D_2\right)-\frac{l(l+1)}{4}\left(D_0+D_1+D_2\right)\right) \tag{55}$$

$$\beta = \frac{1}{4} - \left(\frac{\mu}{2\alpha^2\hbar^2\delta_2}\left(v_0\delta_1 + \delta_2\delta_3\alpha^2 D_0\right) - \frac{D_0 l(l+1)}{4} \right)$$
$$- \left(-\frac{\mu v_0\delta_1}{2\alpha^2\hbar^2\delta_2} - \frac{\mu\delta_3}{2\hbar^2}\left(2D_0 + D_1\right) - \frac{l(l+1)}{2}\left(D_0 + \frac{D_1}{2}\right) \right)$$
$$- \left(\frac{\mu\delta_3}{2\alpha\hbar^2}\left(\alpha D_0 + D_1 - \alpha D_2\right) - \frac{l(l+1)}{4}\left(D_0 + D_1 + D_2\right) \right) \tag{56}$$

Therefore equation (53) reduce to

$$-\frac{\mu E_{nl}}{2\alpha^2\hbar^2} = \eta + \left[\frac{\gamma - \left[\left(n + \frac{1}{2}\right) + \sqrt{\beta}\right]^2}{\left\{\left(n + \frac{1}{2}\right) + \sqrt{\beta}\right\}} \right]^2 \tag{57}$$

$$E_{nl} = -\frac{2\alpha^2\hbar^2}{\mu}\eta - \frac{2\alpha^2\hbar^2}{\mu}\left[\frac{\gamma - \left[\left(n + \frac{1}{2}\right) + \sqrt{\beta}\right]^2}{\left\{\left(n + \frac{1}{2}\right) + \sqrt{\beta}\right\}} \right]^2 \tag{58}$$

Simplifying and rearrange equation (55), (56) and substitute into (58) we obtain the energy spectrum

$$E_{nl} = -\frac{2\alpha^2\hbar^2}{\mu}\left(\frac{\mu\delta_3}{2\alpha\hbar^2}\left(\alpha D_0 + D_1 - \alpha D_2\right) - \frac{l(l+1)}{4}\left(D_0 + D_1 + D_2\right) \right)$$
$$- \frac{2\alpha^2\hbar^2}{\mu}\left[\frac{ \begin{array}{c} \frac{\mu v_0\delta_1}{2\alpha^2\hbar^2\delta_2} + \frac{\mu\delta_3}{2\alpha\hbar^2}\left(\alpha D_1 - D_1 + 2\alpha D_2\right) + \frac{l(l+1)}{2}\left(2D_0 + 3D_1 + D_2\right) \\ -\left[\left(n + \frac{1}{2}\right) + \sqrt{\frac{1}{4} - \frac{\mu v_0\delta_1}{\alpha^2\hbar^2\delta_2} + \frac{\mu\delta_3}{2\alpha\hbar^2}\left(\alpha D_0 - D_1 + \alpha D_2\right) + \frac{l(l+1)}{4}\left(4D_0 + 2D_1 + D_2\right)}\right]^2 \end{array} }{ \left\{\left(n + \frac{1}{2}\right) + \sqrt{\frac{1}{4} - \frac{\mu v_0\delta_1}{\alpha^2\hbar^2\delta_2} + \frac{\mu\delta_3}{2\alpha\hbar^2}\left(\alpha D_0 - D_1 + \alpha D_2\right) + \frac{l(l+1)}{4}\left(4D_0 + 2D_1 + D_2\right)}\right\} } \right]^2 \tag{59}$$

$$E_{nl} = -\alpha\delta_3(\alpha D_0 + D_1 - \alpha D_2) + \frac{l(l+1)\alpha^2\hbar^2}{2\mu}(D_0 + D_1 + D_2)$$
$$-\frac{2\alpha^2\hbar^2}{\mu}\left[\frac{\frac{\mu v_0\delta_1}{2\alpha^2\hbar^2\delta_2} + \frac{\mu\delta_3}{2\alpha\hbar^2}(\alpha D_1 - D_1 + 2\alpha D_2) + \frac{l(l+1)}{2}(2D_0 + 3D_1 + D_2) - \left[\left(n+\frac{1}{2}\right) + \sqrt{\frac{1}{4} - \frac{\mu v_0\delta_1}{\alpha^2\hbar^2\delta_2} + \frac{\mu\delta_3}{2\alpha\hbar^2}(\alpha D_0 - D_1 + \alpha D_2) + \frac{l(l+1)}{4}(4D_0 + 2D_1 + D_2)}\right]^2}{\left\{\left(n+\frac{1}{2}\right) + \sqrt{\frac{1}{4} - \frac{\mu v_0\delta_1}{\alpha^2\hbar^2\delta_2} + \frac{\mu\delta_3}{2\alpha\hbar^2}(\alpha D_0 - D_1 + \alpha D_2) + \frac{l(l+1)}{4}(4D_0 + 2D_1 + D_2)}\right\}}\right]^2 \quad (60)$$

Equation (60) is the energy spectrum for Hylleraas plus inversely quadratic potential and is similar to that obtained by Ituenet al. (Ituen et al., 2015).

To obtain the eigenfunction or wave function, we consider the weight function $\rho(s)$ giving from equation (29) and using the values from equation (50) yield

$$\rho(s) = s^{1+2\sqrt{\varepsilon^2 - C} - 1}(1-s)^{2+2\left\{\sqrt{\frac{1}{4} - A - B - C} + \sqrt{\varepsilon^2 - C}\right\} - \left(1 + 2\sqrt{\varepsilon^2 - C}\right) - 1} \quad (61)$$

$$\rho(s) = s^{2\sqrt{\varepsilon^2 - C}}(1-s)^{2\sqrt{\frac{1}{4} - A - B - C}} \quad (62)$$

The wave functions is gotten by using equation (30)

$$\chi(s) = P_n^{\left(2\sqrt{\varepsilon^2 - C},\, 2\sqrt{\frac{1}{4} - A - B - C}\right)}(1-2s) \quad (63)$$

where, $\in = 2\sqrt{\varepsilon^2 - C}$, and $\mathrm{H} = 2\sqrt{\frac{1}{4} - A - B - C}$

$P_n^{(\in,\mathrm{H})}$, is the Jacobi polynomial.The second part of wave functions is gotten by using equation (32)

$$\varphi(s) = s^{\frac{\in}{2}}(1-s)^{\frac{1+\mathrm{H}}{2}} \quad (64)$$

Therefore, the total wave function is obtained by using equation (63) and (64) into Equation (34)

$$\psi_{nl}(s) = N_{nl}s^{\frac{\in}{2}}(1-s)^{\frac{1+\mathrm{H}}{2}} \times P_n^{(\in,\mathrm{H})}(1-2s) \quad (65)$$

where, N_{nl} is the normalization constant, which can be evaluating using normalization condition

$$\int_0^\infty \psi_{nl}^*(s)\psi_{nl}(s)ds = 1 \quad (66)$$

4. Conslussion

We used parametric form of the Nikiforov-Uvarov methodto obtained bound state solution to Hylleraas plus inversely quadratic potential using Schrodinger equation. By using the elegant approximation for the centrifugal term, we obtain approximately the energy eigenvalues and the unnormalized wave function expressed in terms of the Jacobi polynomials for arbitrary wave states.

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