

Foundational Problems of Quantum Theory: Novel Approach to Temporal Probability Density

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Abstract: Present study focuses on some foundational problems of quantum theory specifically deals with the concept of probability density and relating introductory problems. In this sense, the work initially investigates the origins of the general probability theory and re-examines the concepts of spatial and temporal probability densities based on genuine epistemological and ontological arguments. In order to tackle the foundational problems, standard theory is primarily memorised and criticized scientifically and philosophically in terms of foundationally disappearing term of time dependent potential energy within the time and space dependent Schrödinger wave equation. Based on those arguments, the problematic inconsistency between the spatial and temporal probability density functions is underlined. Given the problem, an original approach previously suggested, is concisely described and extended to resolve the existing problem. The novel approach, based on a novel time dependent Schrödinger wave equation, resolves the discrepancy with the classical wave equation and also leads to time dependent temporal probability densities even for the time free potential energies. Novel temporal probability density function is also normalized and has a fluctuation period of around 10^{-16} s which is very short compared to the atomic time scales.

Keyword: Foundations of quantum theory, probability density, temporal probability density, quantum dynamics, quantum philosophy.

1. Introduction

Quantum theory, in spite of outstanding achievements, surprisingly faces some foundational complications such as probability density, quantization, meaning of the wave function, de Broglie's wave-particle duality and Heisenberg's uncertainty principle in addition to time evolution of quantum systems (Kleppner et al., 2000; Temark et al., 2001; Leifer, 2014; Davies et al., 1993; Omnes, 2018). Inherent paradoxes have additionally led to some disparagements, such as Einstein, Podolsky, Rosen (EPR) Paradox or Quantum Entanglement (Einstein et al., 1935; Horodecki et al., 2009), Quantum Decoherence (Zurek, 2003) and Quantum Zeno Effect (Misra et al., 1977), all to some extent challenge the theory both scientifically and philosophically. Therefore, quantum theory persistently faces certain amount of criticism and counterintuitive results due mostly to foundational problems and famous debate between Einstein and Bohr does not seem to be terminated and no consensus about essential questions about probabilistic

structure of the quantum theory has been reached by the scientific community (Schlosshauer et al., 2013; Sommer, 2013).

Quantum theory was founded on the assumptions that time and space are *infinitely continuous*, which is rather questionable and completely *independent variables* and also quantum particles are *unbreakable solid spheres* and their motion are characterised by matter/Schrödinger waves (Von Neumann, 2018). Consequently, physical existence of a quantum particle and also dynamical variables such as velocity, momentum, energy etc. are all determined by those spatially and temporally distributed Schrödinger waves. Therefore, motion of any quantum particle is fully governed by spatially and temporally distributed matter waves and the wave nature of the quantum motion naturally leads to extra difficulties (Griffiths et al., 2018). Especially immeasurability of the actual wave functions stands as a fundamental problem of the theory and the physical link between the quantum theory and the real atomic world was set by defining the probability density function foundationally suggested by Born (Born, 1955). Hence, any set of experimental measurements can only be linked to the theory statistically through the well-known concept of probability density function.

Probability theory naturally plays a central role in understanding and interpreting the probability density functions in quantum mechanics. The probability concept, based on the standard probability theory, originates from firstly *insufficient information* about the deterministic physical laws or mechanisms or/and secondly *inability of controlling external mechanisms* and parameters that influence the specific measurement/outcome. Thus, it is essential to go deeper and re-examine the origins of quantum probability density functions in terms of the standard probability theory (Erol, 2020).

Mathematical foundations of quantum theory are also based on two distinct operators, firstly time dependent Hamiltonian operator, $\hat{H} = i \hbar \frac{\partial}{\partial t}$, which by definition has no separable potential or kinetic energy components and secondly space dependent linear momentum operator, $\hat{p}_x = -i \hbar \frac{\partial}{\partial x}$, which leads to the space dependent kinetic energy and consequently space dependent Hamiltonian operator. Space dependent Hamiltonian and consequently Space Dependent Schrödinger Wave Equation (SDSWE) governs space dependent Schrödinger waves with the influence of the position dependent potential energy term, $V(r)$. Similarly, time evolution of the wave functions/state vectors is governed by the well-known Time Dependent Schrödinger Wave Equation (TDSWE), however, problematically for only systems with time free potential energies, $V(r, t) = V(r)$. Nevertheless, due to not including time dependent potential energy term, $V(t)$, time dependent Schrödinger equation cannot possibly be taking into account the influence of the time dependent potential energies. This fundamental and foundational problem was later partially resolved by developing the well-known Perturbation Theory which considers the time dependent potential energies, $V(t)$, as small perturbations over the total mechanical energy, E (Griffiths, 2018).

Fundamental outcomes of the standard quantum theory point out that quantum systems having time free potential energies, $V(r, t) = V(r)$, demonstrate strongly position dependent probability densities, $P(r)$, however time free temporal probability densities,

$P(t) = \text{constant}$. Nevertheless, given the time dependent potential energies, $V(r, t)$, temporal probability densities turn out to be dynamic, $P(t)$, similar to the spatial probability density. This fundamental and important issue seems to be effectively ignored by the founders of quantum mechanics. The fundamental question at this point is; *how is it possible that a single quantum particle could be having a space probability density continuously changing while the time probability density stays uniform?* Ontologically and epistemologically speaking, if and when a probability density is dynamic in space, it must be dynamic in time. This foundational problem, to our view, originates from the order of the governing differential equations and tackled previously by the author Erol (Erol, 2020). TDSWE basically governs the time evolution of the Schrödinger waves and because of being a first order differential equation it leads to uniform temporal probability densities, in other words, it leads to stationary states. The SDSWE, on the other hand, is a second order differential equation and leads to continuously varying space probability densities. The concept of time is, on its own, a complicated concept in the sense that genuinely speaking, it is not reversible, not controllable and not stoppable. Unlike the spatial issues, quantum particles cannot be confined in time or their motion cannot be limited to a certain time period by setting some boundary conditions (Griffiths, 2018). Ontological and epistemological approaches to the subject of matter forward that *a quantum particle must exist at a specific point at a particular time* therefore the actual existence probability densities ought to be the same in character in space and time. In other words, should the existence probability density vary with position, for the same quantum particle, ought to change with time as well.

Foundational complexities and problems of quantum theory are being continuously debated and no resolutions have been achieved so far on many points. In order to obtain a full and complete picture of the atomic world, alternative approaches should unquestionably be developed. Therefore, it seems quite legitimate to re-consider the foundations of the quantum theory both philosophically and scientifically to search for answers for particular conceptual difficulties and paradoxes as well as searching for the possibility of some alternative approaches relating to the lack of some obvious components (Zeilinger, 1999; Wallace, 2001; Schlosshauer, 2005). Present work, therefore, initially and philosophically questions present standard quantum theory in terms of spatial and temporal probability densities of the quantum particles. Clear discrepancy between spatial and temporal existence probability densities of the same quantum particle is tackled and re-examined. A novel approach is underlined and described in order to overcome the discrepancy and certain ontological and epistemological problems. The novel approach suggested also leads to the undiscovered field of temporal probability densities of quantum particles.

2. The Concepts of Probability and Existence Probability Density

The concept of probability, in general, is valid for the situations for which outcome of any specific measurements/event cannot be known before the actual event occurs (Kolmogorov, 1933). Therefore, the outcome of any probabilistic event is assumed to be determined randomly/stochastically by chance. Probability theory and therefore

incapability of predicting any specific outcome, in fact, originates from firstly *lack of knowledge* that fully governs the outcome and/or secondly *inability to control environmental effects* that influence the specific outcome (Jaynes, 2003). The first term of *lack of knowledge* means specific/deterministic physical laws or governing parameters are not known and the second term of *uncontrollable environmental effects* means the governing parameters or physical laws are known however cannot be fully controlled. In other words, if one knew the deterministic physical laws and additionally if one could fully control the environmental effects then no probabilistic structure could be appearing.

Classical mechanics is obviously deterministic in character hence the outcome of any specific event can perfectly be predicted beforehand. However, for every individual measurement, environmental effects cannot be fully controlled so the outcome of the measurement may lead to a negligible difference which is only the problem of the measurement system. Quantum mechanics is, on the other hand, foundationally probabilistic theory which, based on the origins of probability theory, means firstly deterministic physical laws or governing parameters are not discovered yet or/and secondly environmental effects that determine the outcome of any specific measurement/event cannot be controlled. This is why quantum theory ought to be handled very carefully and the foundations of the theory must be re-examined and studied with exceptional attention (Afshar et al., 2007).

In order to resolve the spatial and temporal probability functions for a general physical case, it is considered that a particle, under full control of deterministic physical laws, relentlessly moves within the spatial distance of L and within an overall time period of T . Additionally, it is assumed that this particle moves a position interval of dx within the time interval of dt . Then existence probability within the time interval of dt can be defined as $p=dt/T$ and similarly existence probability within the position interval of dx can be defined as $p=dx/L$. At this stage, for the same particle and for the same motion, spatial existence probability density can be defined as probability per unit length, in 1 dimension, which can mathematically be modelled as,

$$P(x) = \frac{dt/T}{dx} \quad (1)$$

For the same particle and situation, temporal probability density which can similarly be defined as existence probability per unit time, can be defined formulised as,

$$P(t) = \frac{dx/L}{dt} \quad (2)$$

Obviously, by definition, the sum of the probability densities must be equal to unity for the overall motion, that is the normalisation of the probability functions. Therefore for the case described above, both temporal probability density, $P(t)$, and spatial probability density, $P(x)$, must be normalised by the following equations, $P= \int_0^T P(t) dt = 1$ and $P= \int_0^L P(x) dx = 1$.

The basic approach described above clearly demonstrates that any particle existing in space must also be appearing in time. In other words, if the physical particle is in motion in space must be moving in time too. Therefore, if and when the spatial probability density

changes, for the same particle, temporal probability density must be changing in accordance with the equations (1) and (2).

Probabilistic structure of quantum mechanics is problematic in two ways: Firstly, quantum theory productively resolves highly complicated behaviour of particles or ensemble of particles at atomic scales by assuming that the quantum particles are unbreakable objects and accompanied by physical waves within the space-time geometry which means that the quantum particles relentlessly move in space-time geometry like waves or in wave character (Afshar et al., 2007; Yang, 2005). Resolution of the motion of a single atomic particle by means of physical waves inevitably introduces certain complications and interpretation problems since physical waves spread through space and time however a single particle does not (Rauch et al., 2015). In this sense, it is quite demanding to understand and visualise the case physically and also philosophically. The second fundamental problem arises from the immeasurability of the actual wave functions and consequently lack of connection between the real world and quantum theory. How can a single particle can be moving with wave nature in space-time geometry is very confusing and tackling for the majority of physicists (Schlosshauer, et al., 2013; Sommer, 2013). The probabilistic structure of quantum theory, to our view, clearly originates from the wave nature of the motion of the quantum particles.

The general definitions of temporal and spatial probability densities and the physical case described above can surely be employed for the quantum particles. Following scheme is standard in quantum mechanics in extracting the probability densities (Griffiths et al. 2018). It is well-known that the physical concepts or so-called dynamic variables in quantum mechanics are characterised by operators. The operators subsequently determine the wave functions through the well-known Eigen value- Eigen function equation, that is simply written as, $\hat{A} \Psi(x, t) = a \Psi(x, t)$. The wave character of the motion of the particles is described by the mathematical wave function, $\Psi(x, t)$, called as the Eigen function of the operator \hat{A} and a denotes the Eigenvalue of the operator. The wave function, $\Psi(x, t)$, mathematically describes time and space evolutions of the physical waves accompanying the quantum particle and is extracted from a deterministic equation. The expression above is also very important in the sense that it connects the theory and measurement/real world. The equation more clearly means that if one measures the dynamical variable or operator \hat{A} , the Eigenvalue of “a” should be obtained. Application of time and space dependent total mechanical energy operator, Hamiltonian operator, to the equation which means $\hat{A} = \hat{H}$, leads to the well-known Time and Space Dependent Schrödinger Wave Equation (TSDSWE) which in 1 dimension is formulated by,

$$i \hbar \frac{\partial}{\partial t} \Psi(x, t) = \left[\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \Psi(x, t) \quad (3)$$

where \hbar denotes reduced Planck's constant, i denotes the complex number operator, m denotes the mass of the particle, $V(x)$ denotes the only space dependent potential energy of the particle and finally $\Psi(x, t)$ denotes the space and time dependent wave function. Please problematically note at this point that space and time dependent wave function, $\Psi(x, t)$, is governed by only space dependent potential energy, $V(x)$. The solution of TSDSWE, $\Psi(x, t)$, smoothly and deterministically leads to mathematical

expressions of the physical waves governing the motion of the quantum particle. However, the wave function itself is not a measurable quantity since it accompanies the quantum particle hence physical connection to the real/physical world is not possible as it stands. This fundamental problem is famously tackled and resolved by the definition of the Born who suggested and introduced the concept of probability as the existence probability density symbolised by $P(x, t)$ and defined as $P(x, t) = \Psi^*(x, t)\Psi(x, t)$ (Born, 1955). It is clear that the probabilistic structure of the quantum theory initiates at this point from the wave nature of the motion of the quantum particles. Assuming that space and time as independent variables one can substitute the expression of $\Psi(x, t) = \Psi(x)\Psi(t)$, then TSDSWE can obviously be separated into two parts, namely only space dependent part and only time dependent part. Space Dependent Schrödinger Wave Equation (SDSWE) in one dimension is then simply given by, $\frac{\partial^2 \Psi(x)}{\partial x^2} + \frac{2m}{\hbar^2} [E - V(x)]\Psi(x) = 0$, which is identical to the classical counterpart and is a second order differential equation. The solution of the SDSWE gives only space dependent wave functions, $\Psi(x)$, which clearly leads to space dependent probability densities, $P(x) = \Psi^*(x)\Psi(x)$, which has equivalent meaning with the equation (1). The second part of the TSDSWE is given by, $\frac{\partial \Psi(t)}{\partial t} + i\omega\Psi(t) = 0$, which is known as time dependent Schrödinger Wave Equation (TDSWE). This equation is very interesting in the sense that *it clearly contradicts with the classical counterpart* and basically gives only the time dependent wave functions, $\Psi(t)$, and temporal probability density $P(t) = \Psi^*(t)\Psi(t)$, which is equivalent to the equation (2).

It is obvious that the wave nature of the motion of the quantum particles leads to the definition of the existence probability density for the motion, namely time dependent and space dependent probability densities (Born, 1955). Ontologically, any single quantum particle existing and relentlessly moving in space and time must continuously be existing in space-time geometry. Epistemologically, that quantum particle relentlessly moving in space-time geometry, ought to have definable temporal and spatial probability densities via the equations of (1) and (2) and varying as a function of time and space. At this stage, space and time are assumed to be no discrete and completely independent variables and also both considered to be unified concepts which means one cannot appear without the other (Saunders, 1998; Feynman, 2005). Nevertheless, standard quantum theory seems to be problematic on this matter because considering quantum particles having time independent potential energies, the theory concludes that the spatial probability densities are strongly *position dependent*, in contrast temporal probability densities are *time free*. Philosophically speaking and considering a single quantum particle, the physical existence of the same particle must be measurable in time and space, moreover if the physical existence of a particle varies with position it must also be varying with time unless the velocity of the particle approaches infinity.

3. Spatial Existence Probability Density

In order to underline certain foundational problems of quantum theory, it is legitimate to discuss the spatial probability density for the quantum particles based on the present

standard theory. The simplest nevertheless a bit of a problematic case of the application of the SDSWE is the free particle situation. Space dependent potential energy of a particle freely moving in space is obviously zero, $V(x) = 0$, then the expression of, $k^2 = \frac{2mE}{\hbar^2}$, within the SDSWE is apparently uniform and expresses the accompanying waves via, $k = \frac{2\pi}{\lambda}$. In this case, general solution of the SDSWE, in exponential form, is written by, $\Psi(x) = Ae^{ikx} + Be^{-ikx}$, where A and B are generally complex constants and determined by the boundary conditions. Using Euler's transformation and arranging the general equation leads to, $\Psi(x) = (A + B)\cos(kx) + i(A - B)\sin(kx)$. Based on this equation, if one calculates the most general form of the spatial existence probability density by using the basic definition of Born, $P(x) = \Psi^*(x)\Psi(x)$; it is straightforward to get,

$$P(x) = |A + B|^2\cos^2(kx) + |A - B|^2\sin^2(kx) \quad (4)$$

It is clear from the expression that the spatial probability density is sinusoidal and varies periodically even though the potential energy and therefore the overall mechanical energy of the quantum particle is considered to be zero, $V(x)=0$. Position dependent probability density, in fact, means that the velocity of the particle should be changing throughout, more specifically if a quantum particle is less likely to be at a point x , then the particle ought to be moving faster and where the existence probability is high then the velocity of the particle ought to be smaller in accordance with the equation (1). Consequently, it can be hypothesised that the spatial probability of the quantum particle changes as a result of the change in the velocity. However, this ordinary approach is also counterintuitive because any change in the probability density should be caused by the change in the velocity and consequently change in the kinetic and indeed overall mechanical energy ($V(x)=0$), which basically violates the conservation of energy. The other important point, of course, is that the spatial probability density cannot be normalised since the particle can freely move to the infinity in space. This fundamental and basic problem actually indicates certain foundational predicaments concerning the present theory.

In order to further underline some basic properties of the spatial existence probability density, infinite potential well example is used as one of the most fundamental model problems of quantum physics. It is straightforward standard activity to resolve the quantised wave functions as follows (Merzbacher, 1970). Considering the infinite potential well, potential energy of the particle is considered to be zero, $V(x) = 0$ inside the well between $x=0$ and $x=L$, and outside of the well the potential energy goes to infinity, $V(x) = \infty$. In this case, the solution of the SDSWE and application of the boundary conditions leads to the result of $\Psi(x) = C\sin(kx)$, where $C = i2A$ can be written. Additionally, the boundary condition of, $\Psi(L) = 0$, leads to the quantisation through, $k_n = \frac{n\pi}{L}$ and normalisation of the wave function, $P = \int_0^L \Psi^*(x)\Psi(x) dx = 1$, also gives $C = \sqrt{\frac{2}{L}}$. The normalised and quantised specific wave functions are then

expressed by, $\Psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$. Based on this quantised solution, the spatial probability of a particle for the quantum state of n is given by,

$$P_n(x) = \frac{2}{L} \sin^2\left(\frac{n\pi}{L}x\right) \quad (5)$$

The final expression clearly shows that even though the quantum particle has no potential energy the spatial existence probability is position dependent. The plots of the spatial probability densities within the infinite potential well as a function of position x , for the states of $n = 1, 2, 3$ are given in the figure 1.

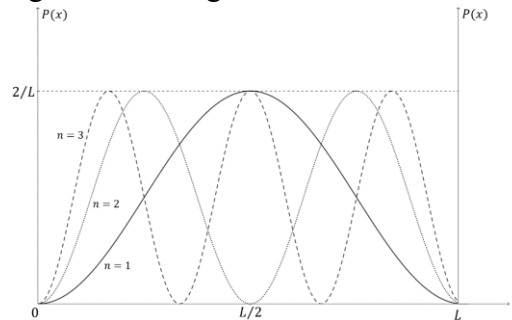


Figure 1. Graphs of the spatial probability densities of finding a particle in the infinite potential well for the quantum states of $n = 1, 2,$ and 3 .

In order to estimate the typical wavelength of the probability density waves, it is straightforward to use the spatial period of the first quantum state, that is $\lambda = L \approx 10^{-10}m$. Therefore, spatial probability density variation should be a definitive property at atomic scales and a very decisive concept for quantum theory. Following general problematic points concerning the space dependent probability density ought to be underlined on the basis of the elementary discussions summarised above. (1) The existence probability density of the free particles strongly varies with position indicating spatially dynamic states in spite of not having time or position dependent potential energies. (2) The standard quantum theory cannot normalise the spatial wave function for the free particles indicating some foundational problems of the theory. (3) The existence probability density of the confined particles, specifically quantum particles in infinite quantum well, also strongly varies as a function of the position. Those specific points given above are going to be tackled and tried to be resolved hereafter.

4. Temporal Probability Density

In general the motion of quantum particles, in space-time geometry, ought to be defined by both space dependent, $\Psi(x)$, and time dependent, $\Psi(t)$, wave functions [26]. Standard quantum theory resolves time dependence of the wave functions initially by considering only the cases for which potential energy is time free. Hence, potential energy of the quantum particle is expressed by $V=V(x)$, not by $V(x, t)$. In order to obtain $\Psi(t)$, obviously only time dependent Schrödinger wave equation that is, $i \hbar \frac{\partial}{\partial t} \Psi(t) = E\Psi(t)$, ought to be solved and by doing so, general solution can be expressed as, $\Psi(t) = \Psi_0 e^{-i\omega t}$, where Ψ_0 denotes the initial value of the time dependent wave function. In order to obtain time dependent probability density for the free particle cases, basic expression of Born,

$P(t) = \Psi^*(t)\Psi(t) = |\Psi(t)|^2$, is employed. Then, temporal probability density for the free quantum particles can be found as,

$$P(t) = |\Psi_0|^2 \quad (6)$$

Elementary calculation, for the quantum particles having time free potential energies, leads to the temporal probability densities to be uniform. It is interesting to note, at this stage, that for the same particle, the spatial probability density was found to be strongly position dependent. To resolve a bit deeper one can employ the general wave functions, given by the well-known superposition principle. According to the superposition principle, general form of the time and space dependent Schrödinger wave function is given by, $\Psi(x, t) = \sum_{n=1}^{\infty} c_n \Psi_n(x) e^{-i\omega_n t}$. In the equation Ψ_0 is inserted into the coefficient c_n . Employing Born's rule to the general wave function leads to the time and space dependent probability density function, that is,

$$P(x, t) = \Psi^*(x, t)\Psi(x, t) = \sum_{n=1}^{\infty} |\Psi_n(x)|^2 |c_n|^2 \quad (7)$$

where the first term, $|\Psi_n(x)|^2$, represents the spatial probability density for the quantum state of n and has clear spatial dependence and the second term, $|c_n|^2$, represents corresponding temporal probability density which is quite interestingly time independent. This clear result epistemologically contradicts with each other and in fact originates from the foundational approaches. Fundamental question at this stage is: Considering a quantum particle and given the spatial probability density varies with position, how can possibly be the temporal probability density uniform?

This basic discussion can also be extended to the more general case in classical physics. Any wave motion, in nature, is governed by the well-known classical time and space dependent wave equation, that is, $\frac{\partial^2}{\partial x^2} \Psi(x, t) = \frac{1}{\vartheta^2} \frac{\partial^2}{\partial t^2} \Psi(x, t)$. General solution of this equation is given by, $\Psi(x, t) = A e^{i(kx - \omega t)} + B e^{-i(kx - \omega t)}$. It is evidently seen from this expression that, in a wave motion, spatial changes are simultaneously accompanied by the temporal changes. As a result, there seems to be some problems with the foundations of quantum physics. More pronouncedly, the fundamental conclusion of standard quantum theory, which is that while spatial probability density of a particle changes its temporal probability density does not change, needs to be re-examined and resolved in a deeper manner. In summary, following fundamental issues seem to be problematic concerning the foundations of quantum theory and ought to be resolved and clarified. (1) Present TDSWE is a first order differential equation, in contradiction with the classical counterpart, and ought to be handled and re-examined. (2) Temporal probability density is uniform for any quantum particle having time free potential energies which contradicts with the spatial counterpart. (3) Temporal probability density cannot be normalised even though the particle cannot be confined or restricted in time.

5. Novel Approach to Temporal Probability Density

Quantum theory resolves time dependence of the wave functions and consequently temporal existence probability densities in two steps. The first step deals only with the cases for which the potential energy of the particle is completely time free, hence overall potential energy is given by only the space dependent term, that is $V(r, t) = V(r)$. The

second step deals with the time dependent potential energies, by employing the well-known time dependent perturbation theory by considering time dependence of the potential energy, $V(t)$, as a small term compared to the overall mechanical energy of the particle, E . In fact, foundational problems to our view start from this point by only considering time free potential energies in deriving time end space dependent Schrödinger wave equation, that is the equation (3).

This specific foundational problem recently tackled and resolved by Erol (2020) and explained in detail (Erol, 2020). The problem is resolved by initially defining a novel time dependent linear momentum operator, $\hat{P}_t = -i \hbar \left[\frac{\partial}{\partial t} \frac{\partial t}{\partial x} \vec{i} + \frac{\partial}{\partial t} \frac{\partial t}{\partial y} \vec{j} + \frac{\partial}{\partial t} \frac{\partial t}{\partial z} \vec{k} \right]$ and following that a novel time dependent Schrödinger wave equation, (NTDSWE) is derived as,

$$\frac{\partial^2 \Psi(t)}{\partial t^2} + \left[\frac{4}{9\hbar^2} [E - V(t)]^2 - i \frac{2}{3\hbar} \frac{\partial V(t)}{\partial t} \right] \Psi(t) = 0 \quad (8)$$

The derived NTDSWE is obviously identical to the classical counterpart with the consideration of $w^2(t) = \frac{4}{9\hbar^2} [E - V(t)]^2 - i \frac{2}{3\hbar} \frac{\partial V(t)}{\partial t}$, and smoothly resolving the apparent conflict. In the expression, $w(t)$ denotes the time dependent angular frequency of the Schrödinger waves, critically $V(t)$ denotes the time dependent potential energy and E denotes the total mechanical energy. NTDSWE obviously has some original and constructive properties. Firstly, it is a second order differential equation and secondly it is identical to the classical counterpart and thirdly it contains directly the term of time dependent potential energy, $V(t)$. Hence, the novel equation has no contradiction with the present quantum theory and additionally it resolves some of the foundational problems. It is interesting to note that the angular frequency is a complex function of t and can be resolved accordingly. Henceforth, it is possible to describe the squared angular frequency in the form of, $w^2(t) = A - iB$, where A and B are relevant parameters of the motion and can be expressed by $A = \frac{4}{9\hbar^2} [E - V(t)]^2$ and $B = \frac{2}{3\hbar} \frac{\partial V(t)}{\partial t}$. The standard complex analysis proposes that the squared angular frequency can alternatively be defined as $w^2(t) = |Z|e^{-i\alpha(t)}$, where the parameter of $|Z| = \sqrt{A^2 + B^2}$ represents the magnitude and $\alpha(t) = \tan^{-1} \left(\frac{B}{A} \right)$ represents the relevant angle of the complex function. NTDSWE is obviously a second order differential equation, substantially different and more informative compared to the present TDSWE, including instantly varying potential energy term and therefore angular frequency.

In order to obtain exact outcomes of this novel approach, based on previously given clarifications and explanations, approximate temporal wave function acquired from the second order NTDSWE is employed and temporal probability density function, $P(t) = \Psi^*(t)\Psi(t)$, is generally found to be,

$$P(t) = 4F_0^2 |Z|^{-1/2} \cos^2 [\theta(t)] \quad (9)$$

where the angle is given by $\theta(t) = \int w(t)dt$ and openly time dependent through the expressions of $w(t) = |Z|^{1/2} e^{-i\frac{\alpha(t)}{2}}$ and $\alpha(t) = \tan^{-1} \left(\frac{B}{A} \right)$. This expression of, $P(t)$, is the most general form of the time dependent probability density obtained by using the

second order NTDSWE without any restrictions and clearly shows the time dependence. NTDSWE clearly leads to results that are consistent with the present perturbation theory suggesting dynamic quantum states and resolves the apparent conflict.

In order to compare the outcomes of the NTDSWE with the results of the current quantum theory, well-known model problem of infinite quantum well situation is resolved by basically substituting, $V(t)=0$, for the case the particle is in the potential well. In this case, the parameters transform into $A = \frac{4}{9\hbar^2} E^2$, $B=0$, $|Z| = A$ and $\theta = w_0 t$ where $w_0 = \frac{2}{3\hbar} E$. In this case, the temporal probability density, for the infinite quantum well problem, can be given by,

$$P(t) = 4F_0^2 |A|^{-1/2} \cos^2 [w_0 t] \tag{10}$$

where the term of $|A|^{-1/2} = \frac{3\hbar}{2E}$ is constant and the expression is time dependent only through the $\cos^2 [w_0 t]$ expression. At this stage it seems legitimate to consider the apparent spatial quantisation of the overall mechanical energy. It is well-known fact that the spatial states are quantised due to the confinement of the particle within the well between $x=0$ and $x=L$. The overall mechanical energy quantisation is expressed by $E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$. The energy quantisation ought to be influencing the vibrational modes of the quantum particle through the famous energy equation of Planck and by employing the expression of $E_n = \hbar w_n$ one can straightforwardly obtain quantised angular frequency of the states as, $w_n = \frac{2}{3\hbar} E_n = \frac{n^2 \pi^2 \hbar}{3mL^2}$. Hence, temporal probability densities, for the infinite quantum well problem, can be expressed by,

$$P_n(t) = C_n \cos^2 [w_n t] \tag{11}$$

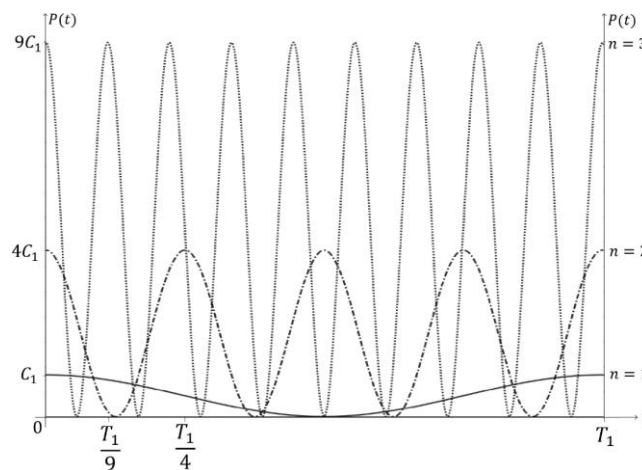


Figure 2. Graphs of the temporal probability densities for the infinite potential well for the quantum states of $n = 1, 2$ and 3 . In the figure, $T_1 = \frac{6mL^2}{\pi\hbar}$ denotes the period of the first quantum state, $T_2 = \frac{T_1}{4}$ and $T_3 = \frac{T_1}{9}$ represent the periods of second and third quantum states respectively.

where $C_n = 4F_0^2 |A_n|^{-1/2}$ denotes the amplitude of the quantised probability density functions for the infinite quantum well with $A_n = \frac{4}{9\hbar^2} E_n^2$. As a result, NTDSWE concludes that the temporal probability density, concerning quantum particles in the infinite quantum well, is clearly time dependent. Additionally, the period of the temporal probability density, for the quantised states, is given by $T_n = \frac{T_1}{n^2}$ where the period for the first quantum state is $T_1 = \frac{6mL^2}{\pi\hbar}$. Temporal probability amplitude of the first quantum state can easily be written by $C_1 = \frac{6F_0^2\hbar}{E_1}$ and consequently the probability amplitudes, for the next quantum states, can be written as, $C_2 = \frac{C_1}{4}$ and $C_3 = \frac{C_1}{9}$. The graphs of the quantised probability densities are illustrated in the figure 2.

The temporal probability density in real terms means that the quantum particles relentlessly moving in space must indeed be moving in time, hence, at any spatial fixed point of r , the particle may coincide at only certain times of the measurement and consequently may exist at other points for the rest of the time. For instance, consider that N simultaneous measurements are being carried out on N identical hypothetical quantum systems for a specific point of r , if the apparatus measures n simultaneous existence at the instant of t , then the existence probability can simply be estimated by, $P = n/N$. Therefore, the quantum particles never disappear in time and the probability density function must mathematically be continuous, just like the spatial case. Based on the definition of the temporal probability density, exact physical meaning of the temporal wave function of $\Psi(t)$ can be expressed as the square root of existence probability per unit time, at a fixed point of r . Therefore, the physical dimension of the temporal wave function is $\Psi(t) = \left[\sqrt{\frac{\text{Probability}}{T}} = T^{-1/2} \right]$.

It is important, at this point, to express that apparent time dependence of the temporal probability densities for quantum particles, concerning the time free potential energies, could not physically be observed so far because the period of the density probability waves is too short to observe. In order to estimate a typical value, one can consider the infinite potential well example and it is straightforward to estimate the time scale or the period of the probability waves for the infinite quantum well, that is, $T_1 = \frac{6mL^2}{\pi\hbar} \approx 10^{-16} \text{ s}$. This time scale is ultra-short and out of the range of current interest and time measurement limits. In order to just compare the typical wavelengths of the space dependent probability density waves, we simply look at the wavelength of the plane waves, that is, $\lambda = L \approx 10^{-10} \text{ m}$. This length scale is just around typical atomic scales hence space probability density variation is a definitive property at atomic scales and a very important concept for quantum theory.

6. Normalisation of Temporal Probability Density

Quantum mechanics, instead of deterministically estimating the outcome of any event or measurement, only offers certain percentages of probability for that specific event or

measurement. The probability here means the existence probability which measures the physical presence percentage of the quantum particle at a certain point of x , at a certain time of t . Spatial probability density by definition refers to the overall existence probability per unit length in 1 dimension, $P(x)$ and per unit volume in 3 dimensions, $P(r)$. Temporal probability density similarly refers to the probability per unit time, $P(t)$. Spatial and temporal joint probability density in 1 dimension, $P(x, t)$, can be defined as the product of spatial and temporal probability densities, $P(x, t) = P(x) P(t)$, which means the existence probability density of the quantum particle at a point of x and at a time of t .

Present theory initially handles the temporal probability densities for the cases in which the quantum particle has completely time free potential energies, $V(x, t) = V(x)$. Present theory leads to following fundamental conclusions; (1) Temporal probability density function is found to be uniform, which means static quantum states and for a certain point of x in 1 dimension the quantum particle can appear with same probability at any time. (2) The present theory cannot estimate the actual percentage of the constant temporal probability density.

The first conclusion of the standard quantum theory is philosophically and scientifically problematic. It is highly counterintuitive to assume that a quantum particle relentlessly moving in both space and time, spatial probability varies however time probability does not change. This result of the present theory violates the philosophical approaches both epistemologically and ontologically. Because, given a single quantum particle characteristic motion of the particle obviously determines the existence probability and any change of the spatial probability density must be accompanied by the change in temporal probability density. Hence this is an open problem that ought to be handled. The second problem arises from the disability of the present theory on the calculation of the constant value of the temporal probability density. This is also a problem in the sense that the temporal probability density cannot be normalised therefore necessary physical meaning ought to be attributed to the particle cannot be managed. Therefore, the present theory has some apparent foundational problems.

The present effort tries to address some of the foundational problems mentioned above. Given a quantum particle existing and moving continuously within time-space geometry, following critical conditions ought to normally be verified by the temporal probability density function; (1) Assuming time and space are totally independent and infinitely no discrete variables then temporal probability density function must be continuous similar to the spatial probability density. This means the quantum particle must be existing at every instant between the creation and annihilation moments. (2) Since spatial and temporal probability densities are defined for the same quantum particle then if the spatial density varies, accordingly temporal probability density ought to be varying and hence cannot be uniform. (3) Temporal probability density must obviously be normalised hence overall existence probability density must be equal to unity.

This effort genuinely meets all the points briefly mentioned above. The first point is about the continuity of the physical existence of the quantum particle. The physical existence obviously must be continuous since the quantum particle cannot disappear in

accordance with the most fundamental physical conservation law that is the conservation of energy/matter.

The other central problem was the uniformity of the temporal probability function even though the spatial probability density for the same particle varies with position. The present effort also solves this problem by offering the equations (9), (10) and (11) which obviously change with time by sinusoidal form. The change in the temporal probability in fact indicates certain variation of the velocity of the quantum particle, which would obviously result in continuously varying spatial and temporal probability densities.

The third point, in quantum mechanics, both temporal and spatial probabilities of the particle must be normalized. Mathematical functions that cannot be normalized cannot normally express the presence of the particles. Spatial normalization condition states that the particle must exist somewhere (or must occupy any position) in all space. Likewise, the temporal normalization condition should state that the particle must exist hypothetically somewhere between $t = -\infty$ and $t = +\infty$. Therefore, concerning a single quantum particle temporal probability density function can be normalised by the equation of, $P = \int_{-\infty}^{+\infty} |\Psi(t)|^2 dt = 1$. However, in line with the inferences made in the previous sections, if the probability of finding a particle in time is examined, it is seen that the probabilities change sinusoidally and periodically. Based on this, it is quite consistent to say that all the physical information about the particle should exist in a period of time. When we think of an object making smooth circular motion, it is clearly understood that the motion of the object is periodic, this object returns to where it started again after a period of time and repeats its movement. Within these repetitions, the object is in a specific position within a period of each period and the probability of finding this object within a period of time ought to be equal to unity. Concerning periodic temporal probability densities, since all the necessary information is present within the first period of time then normalisation can be managed within the first period of time, that is,

$$P_1 = \int_0^{T_1} |\Psi_1(t)|^2 dt = 1 \quad (12)$$

Similarly, it is obvious that the same amount of information is present within the second and third period of time hence for the second period of time and third period of time etc. $P_2 = \int_0^{T_2} |\Psi_2(t)|^2 dt = 1$, $P_3 = \int_0^{T_3} |\Psi_3(t)|^2 dt = 1$. Then overall joint existence probability between $t = -\infty$ and $t + \infty$ can be calculated, based on the standard probability theory, by multiplication of the probability densities for each period, which can be given by,

$$P = P_1 P_2 P_3 \dots P_\infty = 1 \quad (13)$$

The discussions of temporal probability density, based on the NTDSWE, is enlightening and could have significant implications and could open new routes for investigation within current quantum theory. Previously derived NTDSWE here originally leads to the dynamic temporal quantum states not only for the quantum particles having strongly time dependent potential energies, in agreement with the results of the PT, but also for the time free potential energies, in contrast to the TDSWE.

7. Conclusions

The present work has briefly tackled some aspects of the foundational problems of the current quantum theory, specifically investigating the concept of probability density functions. To do so, initially the origins of the standard probability theory is expressed by underlining the insufficient information and uncontrollable environmental effects. Consequently, spatial and temporal probability densities of quantum particles are re-examined based on the present probability theory and on some epistemological and ontological arguments. In this sense, the present quantum theory is mainly criticized on; firstly, not containing time dependent potential energy term within the time dependent Schrödinger wave equation, secondly demonstrating completely time free temporal probability densities in spite of space dependent spatial probability densities. In order to overcome those foundational problems, a novel approach recently suggested by Erol (2020) is extended and employed to resolve the discrepancy between the temporal and spatial probability densities for the free particle and infinite quantum well cases. Novel approach shows no disagreement with the present quantum theory and smoothly resolves the discrepancy between the temporal classical and quantum wave equations and also resolves the discrepancy between the spatial and temporal probability densities. It is also interestingly concluded that temporal probability density is in fact strongly time dependent and can be normalised even for the free particles, nevertheless it has a fluctuation period of around 10^{-16} s, which is too short to influence the fundamental outcomes of quantum theory. The approach suggested is significant in the sense that it simply leads to some new routes and some undiscovered regions of quantum dynamics, ought to be discovered in the future.

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