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Study of Klein Gordon Equation with Minimum Length Effect for Woods-Saxon Potetial using Nikiforov-Uvarov Functional Analysis

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Abstract. The equation of Klein-Gordon for Woods-Saxon potential was discussed in the minimal length effect. We have found the completion of this equation using an approximation by suggesting a new wave function. The Klein-Gordon equation in the minimal-length formalism for the Woods-Saxon potential is reduced to the form of the Schrodinger-like equation. Then the equation was accomplished by Nikiforov-Uvarov Functional Analysis (NUFA) with Pekeris approximation. This technique is applied to gain the radial eigensolutions with chosen exponential-type potential models. The method of NUFA is more compatible by eliminating vanishing the strict mathematical manipulations found in other methods. The energy calculation results showed that angular momentum, quantum number, minimum length parameter, and atomic mass influenced it. The higher the quantum number and angular momentum, the lower the energy. In contrast to the minimum length, the energy spectrum will increase in value when the minimum length parameter is enlarged. An increase in atomic mass also causes energy to increase as the quantum number and angular momentum are held constant.

Keywords: Klein-Gordon equation, minimal length effect, Woods-Saxon potential, Nikiforov-Uvarov Functional Analysis (NUFA) method.

1. Introduction

The equation of Klein-Gordon or Lorentz covariance equation belongs to the relativistic wave equation. This second order equation in time and space has a negative energy solution and a negative probability. This matter made the Klein-Gordon equation not immediately popular the year it was introduced. Accordingly, the Klein-Gordon equation is known for describing the relativistic particles dynamics with zero spin (Lutfuoglu et al., 2018). The Klein-Gordon equation with zero spin is very important in quantum mechanics for explaining complete information about quantum systems. This information can be obtained from the wave function (Badalov et al., 2010).

In 2015, the equation of Klein-Gordon was accomplished for the Woods-Saxon potential (Olgar & Mutaf, 2015). Subsequently, thi equation also has been solved for Eckart potential [4], Kratzer potential [5], and hyperbolic potential (Onate et al., 2017).

The methods usually used to solve the Klein-Godon equation are Asymptotic Iteration Method (Elviyanti et al., 2018; Soeparmi et al., 2018), hypergeometric (Hou & Zhou, 1999; Dianawati et al., 2018), Nikiforov-Uvarov (Ikot et al., 2021), and SUSY (Ikot et al., 2014).

One of the substantial short-range potentials in physics is the Woods-Saxon. This potential is implemented to atomic physics, particle and nuclear physics, and condensed matter. The spherical potential of Woods-Saxon used as the main section of the nuclear structure model succeeded in deducing nuclear energy levels and was used for the interaction of neutrons with heavy nuclei. The potential of Woods-Saxon is also used as part of an optical instance in the elastic scattering of multiple heavy ions with a low energy target (Hamzavi & Rajabi, 2013).

In quantum mechanics, the uncertainty of position and momentum is outlined using the commutation relationship between each operator. The commutation relationship between these two operators is explained in the Heissenberg uncertainty principle (Poojary, 2015).

The principle of Heisenberg's uncertainty in the simultaneous measurement of the position and momentum quantities is that the more certain the result of one quantity is, the more uncertain the result of the measurement of another quantity. Heissenberg's uncertainty principle which is corrected for quantum gravity is called the Generalized Uncertainty Principle (GUP) which is formulated with a minimal length (Alimohammadi & Hassanabadi, 2017). The equation of Klein-Gordon in influence of minimum length has been explained for trigonometric cotangent potential (Cari et al., 2017) and Hulthen potential (Elviyanti et al., 2018).

In this article, we delved into the wave function and energy of the Klein-Gordon with minimum length formalism for Woods-Saxon potential by menas of Nikiforov Uvarov Functional Analysis method. By adjusting new function, the Klein-Gordon equation with minimal length effect is reduced to a Schrodinger-like equation.

The system of this article is as follows. In Chapter 2, the research method is brief, the Woods-Saxon potential and Pekeris approximation are also briefly described in this Chapter. The solutions of the energy spectra and unnormalized wave function are discussed in Chapter 3. Finally, conclusions are presented in Chapter 4.

2. Research Methods

2.1. The esimate solution of Klein-Gordon Equation with minimal length

The uncertainty of position and momentum is an important principle in quantum mechanics. This uncertainty is explained using the commutation relationship between the two operators as follows (Hassanabadi et al., 2017; 19)

$$[\hat{x}, \hat{p}] = ih \tag{1}$$

with $\hat{x}$ represents the operator of position, $\hat{p}$ represents the momentum, i is the imaginary number operator, and $\hbar = h/2\pi$ (h is the constant of Planck). The commutation relationship is shown by the Heisenberg uncertainty principle (Poojary, 2015). This uncertainty principle can be written as follows

$$\Delta x \Delta p_x = \frac{\hbar}{2} \tag{2}$$

The concept of minimum length is introduced as follows

$$\left[\hat{X}, \hat{P}\right] = i\hbar\left(1 + \alpha_{ML}(\Delta P)^2\right) \tag{3}$$

with α_{ML} is the minimum length factor with the interval $0 \le \alpha_{ML} \le 1$. The limit value $\alpha_{ML} \to 0$ and $\alpha_{ML} \to 1$ accordance with normal quantum mechanics and extreme quantum gravity. Operators $\hat{P}$ and $\hat{p}$ represent momentum associated with low and high energy levels.The quantity of momentum for high energies is denoted by $\hat{P}$ and for low energies it is denoted by $\hat{p}$ (Garay, 1994).

Equation (3) is also known as GUP where the parameter α_{ML} have to be calculated from the fundamental theory. The value of α_{ML} will be close to zero when the energy is less than the Planck mass scale with the result that the Heisenberg uncertainty principle is recovered. Equation (3) gives the implied meaning that the minimum length is given as

$$\left[\Delta\hat{X}, \Delta\hat{P}\right] = \frac{\hbar}{2}\left(1 + \alpha_{ML}(\Delta\hat{p})^2\right) \tag{4}$$

The position and momentum operators that are in a commutation relationship with each other can be defined as follows:

$$\hat{X}_i = \hat{x}_i \tag{5}$$

$$\hat{P}_i = \left(1 + \alpha_{ML}\hat{p}^2\right)\hat{p}_i \tag{6}$$

The $\hat{p}$ operator in quantum mechanic is expressed in terms of $\hat{p} = -i\hbar\nabla$, so the quadratic form is (Suparmi et al., 2020)

$$\hat{p}^2 = -\hbar^2\Delta \tag{7}$$

where Δ is the Laplacian operator. The substitution Eq. (7) into Eq. (6), then we get

$$\hat{P}^2 = -\hbar^2\left(1 - 2\alpha_{ML}\hbar^2\Delta\right)\Delta \tag{8}$$

Equation (8) shows the quadratic momentum operator which is affected by the minimum length.

The Klein-Gordon equation is expressed as follows (Chabab et al., 2012; Andaresta et al., 2020)

$$\left(E - V(\hat{r})\right)^2\psi = \left[P^2c^2 + \left(M_oc^2 + S(\hat{r})\right)^2\right]\psi \tag{9}$$

The Klein-Gordon equation with the effect of minimal length is obtained by substituting equation (8) into equation (9), so that

$$\left(E - V(\hat{r})\right)^2\psi = -\left(1 - 2\alpha_{ML}\left(\hbar^2\Delta\right)\right)\left(\hbar^2\Delta\right)c^2\psi + \left(M_oc^2 + S(\hat{r})\right)^2\psi \tag{10}$$

where E is the relativistic energy, $V(\hat{r})$ is a vector component of potential, $S(\hat{r})$ is a scalar component of potential, M_o is rest mass parameter, and P is a three-dimensional linear momentum. By setting V(r) = S(r), $\hbar = 1$ and c = 1, then

$$\left(E^2 - M_o{}^2 - 2(E + M_o)V\right)\psi = -\left(\Delta - 2\alpha_{ML}\Delta^2\right)\psi \tag{11}$$

and here we have set $V(r) \to (1/2)V(r)$. The approximation of the Klein-Gordon equation is gained by using a new wave function as below (Chabab et al., 2016)

$$\psi(r,\theta,\varphi)=(1+2\alpha_{ML}\Delta)\phi(r,\theta,\varphi) \tag{12}$$

The equation (11) becomes

$$\left(-\Delta+4\alpha^{2}{}_{ML}\Delta^{3}\right)\phi(r,\theta,\varphi)-\left(E^{2}-M_{o}{}^{2}-(E+M_{o})V(r)\right)+2\alpha_{ML}\left(E^{2}-M_{o}{}^{2}-(E+M_{o})V(r)\right)\Delta\phi(r,\theta,\varphi)=0 \tag{13}$$

The component of Δ^3 is eliminated because the value $\alpha^{2}{}_{ML}$ is get near to zero and the value of α_{ML} is terribly small. The property showing that Δ is a scalar differential operator has been used in this case. If Δ operates to scalar fields ϕ at a point (r,θ,ϕ), then $\Delta\phi$ will generate in another scalar field. It is known as scalar Laplacian (Elviyanti et al., 2018).

$$\Delta\phi(r,\theta,\varphi)+\left[\left(E^{2}-M_{o}{}^{2}-(E+M_{o})V(r)\right)-2\alpha_{ML}\left(E^{2}-M_{o}{}^{2}-(E+M_{o})V(r)\right)^{2}\right]\phi(r,\theta,\varphi)=0 \tag{14}$$

The spherical Laplacian operator is expressed as

$$\Delta=\frac{1}{r^{2}}\frac{\partial}{\partial r}r^{2}\frac{\partial}{\partial r}+\frac{1}{r^{2}}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^{2}\theta}\frac{\partial^{2}}{\partial\varphi^{2}}\right) \tag{15}$$

By applying Laplacian operator into equation (14) and by supposing the new wave function $\phi(r,\theta,\varphi)=R(r)\Theta(\theta)\Phi(\phi)$, then we have two parts, namely radial and polar of the Klein-Gordon equation. The angular part is given:

$$-\left[\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta}{\partial\theta}+\frac{1}{\Phi}\frac{1}{\sin^{2}\theta}\frac{\partial^{2}\Phi}{\partial\varphi^{2}}\right]=\lambda \tag{16}$$

while the radial part is given:

$$\frac{1}{r^{2}}\frac{\partial}{\partial r}r^{2}\frac{\partial R}{\partial r}+\left[\left(E^{2}-M_{o}{}^{2}-(E+M_{o})V(r)\right)-2\alpha_{ML}\left(E^{2}-M_{o}{}^{2}-(E+M_{o})V(r)\right)^{2}\right]R(r)=\frac{\lambda}{r^{2}}R(r) \tag{17}$$

where λ is a constant variable separation method which accordance in the angular momentum (L). By setting $R(r)=\frac{\chi(r)}{r}$ and $\lambda=L(L+1)$, then

$$\frac{d^{2}\chi(r)}{dr^{2}}-\frac{L(L+1)}{r^{2}}\chi(r)+\left[\begin{matrix}\left(E^{2}-M_{o}{}^{2}-(E+M_{o})V(r)\right)\\-2\alpha_{ML}\left(\left(E^{2}-M_{o}{}^{2}\right)^{2}-2\left(E^{2}-M_{o}{}^{2}\right)(E+M_{o})V(r)+(E+M_{o})^{2}V^{2}(r)\right)\end{matrix}\right]\chi(r)=0 \tag{18}$$

with

$$\begin{aligned}&\gamma=L(L+1);\ o=\left(E^{2}-M_{o}{}^{2}\right)-2\alpha_{ML}\left(E^{2}-M_{o}{}^{2}\right)^{2};\\&\kappa=V_{0}\left(-(E+M_{o})+4\alpha_{ML}\left(E^{2}-M_{o}{}^{2}\right)(E+M_{o})\right);\ \upsilon=2\alpha_{ML}V_{0}{}^{2}(E+M_{o})^{2}\end{aligned} \tag{19}$$

Equation (18) is the equation of Klein-Gordon with minimum length formalism. This equation can be solved for an exponential type potential such as the potential of Woods-Saxon.

2.2. Woods-Saxon potential and Pekeris approximation

The potential of Woods-Saxon is used as the basis for analyzing of the elastic scattering angular distribution of some stable systems (Hamzavi & Rajabi, 2013). This potential also has a fixed geometry (Freitas et al., 2016). The elastic scattering between the nuclei provides great flexibility so as information about nuclear interactions can be obtained. The appropriate solution from the wave equation of this potential provides a conceptual understanding of the resonance and bound states of the interactions between

nuclei (Suparmi et al., 2021). The average field perceived by the valence electrons in the Helium model can be described using the potential of Woods-Saxon (Dudek et al., 2004)

The standard form of Woods-equation is as determined as [3]:

$$V(x) = -\frac{V_0}{1+e^{\frac{r-R}{a}}} \tag{19}$$

where parameter R is defined as the core radius, parameter a represents a shallow layer thickness where the potential is decreased from the value $V = 0$ outside the core to the value $V = -V_0$ inside the core. A simple potential well is obtained by jumping on the surface of the core at $a = 0$.

The equation of Klein-Gordon in the equation (18) for value of $l \neq 0$ cannot be resolved because there is a $1/r^2$ centrifugal factor. We can use a Pekeris approximation method to overcome this. The extension of the term centrifugal in exponential series forms the basis of this approach. This expansion depends on internuclear distance and only considered the second order term, because the next term has a very small value and causes the *l*-dependent potential effectively retains its original shape (Badalov et al., 2010; Feizi et al., 2011), se we get the following notation

$$r = R(x+1) \tag{20}$$

$$\varpi = \frac{R}{a} \tag{21}$$

However, it must be shown that this approach is only in effect for energy states with low vibration. The centrifugal potential is expanded into the Taylor series around $x = 0$ ($r = R$), the centrifugal factor can be approximated as follows

$$f(r) = \frac{\gamma}{r^2} = \frac{\gamma}{R^2}(1+x)^{-2} = \frac{\gamma}{R^2}\left(1-2x+3x^2-4x^3+\ldots\right) \tag{22}$$

We will replace the potential $f(r)$ according to Pekeris approximation, with the expression

$$f(x) = \frac{\gamma}{R^2}\left(d_0 + \frac{d_1}{1+e^{\varpi x}} + \frac{d_2}{\left(1+e^{\varpi x}\right)^2}\right) \tag{23}$$

The parameters d_0, d_1, and d_2 can be determined by expanding $f(x)$ in the Taylor series around the point $x = 0$ ($r = R$)

$$f(x) = \frac{\gamma}{R^2}\left[\left(d_0 + \frac{d_1}{2} + \frac{d_2}{4}\right) - \frac{\varpi}{4}(d_1+d_2)x + \frac{\varpi^2}{16}d_2x^2 + \frac{\varpi^2}{48}(d_1+d_2)x^3 - \ldots\right] \tag{24}$$

We compare equal power of x in Eq. (22) and (24), so we get the constants d_0, d_1, and d_2

$$d_0 = 1 - \frac{4}{\varpi} + \frac{12}{\varpi^2};\ d_1 = \frac{8}{\varpi} + \frac{48}{\varpi^2};\ d_2 = \frac{48}{\varpi^2} \tag{25}$$

Having inserted Eq. (24) into Eq. (18), we obtain

$$\left[\frac{\partial^2}{\partial r^2} + o - \frac{\gamma d_0}{R^2} - \frac{\left(\kappa + \gamma\frac{d_1}{R^2}\right)}{1+e^{\frac{r-R}{a}}} - \frac{\left(\upsilon + \gamma\frac{d_2}{R^2}\right)}{\left(1+e^{\frac{r-R}{a}}\right)^2}\right]\chi(r) = 0 \tag{26}$$

2.3. The method of Nikiforov-Uvarov Functional Analysis (NUFA)

A simple technique applying the NU concept, parameters of NU method, and function analysis method, was proposed by Ikot et al. (2021) to solve second-order differential equations of hypergeometric type. This method is known as Nikiforov-Uvarov Functional Analysis (NUFA). Unlike the NU method which uses polynomial squares and other types that make it more complicated, the NUFA method is more efficient. Once the wave equation is correctly transformed and the singularity is discovered, it will be easy to derive the wave function and energy equation.

The NU method is utilized to solve second-order differential equations in the following form

$$\frac{d^2\psi(s)}{ds^2}+\frac{\tilde{\tau}(s)}{\sigma(s)}\frac{d\psi(s)}{ds}+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0 \tag{27}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, generally second-degree, and $\tilde{\tau}(s)$ is a first-degree plynomial. Then the parametric form of the NU method was introduced (Tezcan & Sever, 2009) as follows:

$$\frac{d^2\psi(s)}{ds^2}+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\frac{d\psi(s)}{ds}+\frac{\psi(s)}{s^2(1-\alpha_3 s)^2}\left[-\xi_1 s^2+\xi_2 s-\xi_3\right]=0 \tag{28}$$

where α_i and $\xi_i(i=1,2,3)$ are all parameters. It can be noticed that the equation (28) has two points that take an infinite value at $s\to 0$ and $s\to\frac{1}{\alpha_3}$, so that we adjust a new wave function as below

$$\psi(s)=s^x(1-\alpha_3 s)^y f(s) \tag{29}$$

By setting the new wave function into Eq. (28), then

$$\begin{aligned}&\left\{-\left[(x+y)^2-(x+y)\right]\alpha_3 f(s)+s(1-\alpha_3 s)f''(s)\right\}-x\alpha_2 f(s)-y\alpha_2 f(s)+\frac{\xi_1}{\alpha_3}f(s)\\&+\left\{\left[\alpha_1+2x-(2x\alpha_3+2y\alpha_3+\alpha_2)s\right]\right\}f'(s)+\left[\frac{\xi_2-\alpha_3\xi_3-\xi_1/\alpha_3}{(1-\alpha_3 s)}-\frac{\xi_3}{s}\right]f(s)\\&+\frac{x(x-1)}{s}f(s)+\frac{y(y-1)}{(1-\alpha_3 s)}\alpha_3 f(s)+\alpha_1\frac{x}{s}f(s)-\frac{y\alpha_3\alpha_1-y\alpha_2}{(1-\alpha_3 s)}f(s)=0\end{aligned} \tag{30}$$

The Gauss hypergeometric equation can be obtained from equation (30) if it complies the following function

$$x(x-1)+x\alpha_1-\xi_3=0 \tag{31}$$

$$\xi_2-\alpha_3\xi_3-\xi_1/\alpha_3-y(\alpha_3\alpha_1-\alpha_2)+y(y-1)\alpha_3=0 \tag{32}$$

We can gain

$$x=\frac{(1-\alpha_1)\pm\sqrt{(1-\alpha_1)^2+4\xi_3}}{2} \tag{33}$$

$$y=\frac{(\alpha_3\alpha_1-\alpha_2+\alpha_3)}{2\alpha_3}\pm\frac{\sqrt{(\alpha_3\alpha_1-\alpha_2+\alpha_3)^2-4\alpha_3(\xi_2-\alpha_3\xi_3-\xi_1/\alpha_3)}}{2\alpha_3} \tag{34}$$

Thus, Eq. (30) becomes

$$\begin{aligned}&s(1-\alpha_3 s)f''(s)+\left\{\left[\alpha_1+2x-(2x\alpha_3+2y\alpha_3+\alpha_2)s\right]\right\}f'(s)\\&-\left\{\alpha_3\left[(x+y)^2-(x+y)\right]+\alpha_3\left(\frac{\alpha_2}{\alpha_3}x+\frac{\alpha_2}{\alpha_3}y-\frac{\xi_1}{\alpha_3^2}\right)\right\}f(s)=0\end{aligned} \tag{35}$$

that gives

$$s(1-\alpha_3 s)f''(s)+\{[\alpha_1+2x-(2x\alpha_3+2y\alpha_3+\alpha_2)s]\}f'(s) - \alpha_3\left\{(x+y)+\frac{1}{2}\left(\frac{\alpha_2}{\alpha_3}-1\right)+\sqrt{\frac{1}{4}\left(\frac{\alpha_2}{\alpha_3}-1\right)^2+\frac{\xi_1}{\alpha_3^{\,2}}}\right\}\left\{(x+y)+\frac{1}{2}\left(\frac{\alpha_2}{\alpha_3}-1\right)-\sqrt{\frac{1}{4}\left(\frac{\alpha_2}{\alpha_3}-1\right)^2+\frac{\xi_1}{\alpha_3^{\,2}}}\right\}f(s)=0 \quad (36)$$

Eq. (35) is a type of hypergeometric of the form

$$z(1-z)\frac{d^2f(z)}{dz^2}+[c+(a+b+1)z]\frac{df(z)}{dz}-[ab]f(z)=0 \quad (37)$$

By adjusting equation (36) and (37), we have the parameters of *a'*, *b'*, and *c'*

$$a'=\sqrt{\alpha_3}\left\{(x+y)+\frac{1}{2}\left(\frac{\alpha_2}{\alpha_3}-1\right)+\sqrt{\frac{1}{4}\left(\frac{\alpha_2}{\alpha_3}-1\right)^2+\frac{\xi_1}{\alpha_3^{\,2}}}\right\} \quad (38)$$

$$b'=\sqrt{\alpha_3}\left\{(x+y)+\frac{1}{2}\left(\frac{\alpha_2}{\alpha_3}-1\right)-\sqrt{\frac{1}{4}\left(\frac{\alpha_2}{\alpha_3}-1\right)^2+\frac{\xi_1}{\alpha_3^{\,2}}}\right\} \quad (39)$$

$$c'=\alpha_1+2x \quad (40)$$

The equation of energy is gained from the last term in equation (36) . Parameters of *a'* or *b'* have negative integer *–n* values, then the function of hypergeometric *f(s)* will be a polynomial of degree *n*. Therefore, the function of hypergeometric verges the quantum state i.e. $a'=-n$, where $n=0,1,2,3,\ldots,n_{\max}$.

$$-n=\sqrt{\alpha_3}\left\{(x+y)+\frac{1}{2}\left(\frac{\alpha_2}{\alpha_3}-1\right)+\sqrt{\frac{1}{4}\left(\frac{\alpha_2}{\alpha_3}-1\right)^2+\frac{\xi_1}{\alpha_3^{\,2}}}\right\} \quad (41)$$

By substituting Eq. (33) and (34) into Eq. (29), the wave function for the NUFA method will be obtained as follows

$$\psi(s)=s^{\frac{(1-\alpha_1)\pm\sqrt{(1-\alpha_1)^2+4\xi_3}}{2}}(1-\alpha_3 s)^{\frac{(\alpha_3\alpha_1-\alpha_2+\alpha_3)}{2\alpha_3}\pm\frac{\sqrt{(\alpha_3\alpha_1-\alpha_2+\alpha_3)^2-4\alpha_3(\xi_2-\alpha_3\xi_3-\xi_1/\alpha_3)}}{2\alpha_3}}\times{}_2F_1(a',b';c';s) \quad (42)$$

3. Result and Discussion

Equation (26) is solved by substituting a new variable,

$$z=\frac{1}{1+e^{\frac{r-R}{a}}} \quad (47)$$

then we have

$$\frac{d^2}{dr^2}=\frac{z^2(1-z)^2}{a^2}\frac{\partial^2}{\partial z^2}+\frac{z(1-z)(1-2z)}{a^2}\frac{\partial}{\partial z} \quad (47)$$

By inserting equation (46) and (47) into equation (26) and multiplied by $\frac{a^2}{z^2(1-z)^2}$, we have

$$\left(\frac{\partial^2}{\partial z^2}+\frac{(1-2z)}{z(1-z)}\frac{\partial}{\partial z}+\frac{a^2}{z^2(1-z)^2}\left(o-\frac{\gamma D_0}{R^2}\right)+\frac{za^2}{z^2(1-z)^2}\left(-\kappa-\gamma\frac{D_1}{R^2}\right)-\frac{z^2a^2}{z^2(1-z)^2}\left(\upsilon+\gamma\frac{D_2}{R^2}\right)\right)\chi(r)=0 \quad (48)$$

By comparing equation (28) and (48), we get

$$\alpha_1=1;\ \alpha_2=2;\ \alpha_3=1$$
$$\xi_1=A^2;\ \xi_2=B^2;\ \xi_3=C^2 \quad (49)$$

with

$$A^2=a^2\left(\upsilon+\gamma\frac{d_2}{R^2}\right);\ B^2=a^2\left(-\kappa-\gamma\frac{d_1}{R^2}\right);\ C^2=a^2\left(o-\frac{\gamma d_0}{R^2}\right) \quad (50)$$

By using Eq. (41) we have the relativistic energy equation

$$-n=(x+y)+\frac{1}{2}+\sqrt{\frac{1}{4}+A^2} \tag{51}$$

By inserting equation (49) to equation (33) and (34) we gain

$$x=\pm C \tag{52}$$

$$y=\pm\sqrt{C^2-B^2+A^2} \tag{53}$$

If we substitute equation (52) and (53) into equation (51), we have

$$C^2=\left(\frac{B^2-A^2}{2n'}+\frac{n'}{2}\right)^2 \tag{54}$$

with

$$n'=n+\frac{1-\sqrt{1+4a^2\left(2\alpha_{ML}V_o{}^2(E+M_o)^2+L(L+1)\frac{D_2}{R^2}\right)}}{2} \tag{55}$$

By using equation (19) and (50) we obtain

$$\left(E^2-M_o{}^2\right)=\frac{1}{a^2}\left(\frac{n'}{2}+\frac{a^2\left(-\frac{L(L+1)}{R^2}(D_1+D_2)-V_o\left(-(E+M_o)-4\alpha_{ML}\left(E^2-M_o{}^2\right)(E+M_o)\right)-2\alpha_{ML}V_o{}^2(E+M_o)^2\right)}{2n'}\right)^2+2\alpha_{ML}\left(E^2-M_o{}^2\right)^2+L(L+1)\frac{D_0}{R^2} \tag{56}$$

By applying equation (52), (53) into equation (29) we get the radial wave function as follows

$$\psi(z)=z^{\pm C}(1-z)^{\pm\sqrt{(C^2-B^2+A^2)}}\times{}_2F_1(a',b';c';z) \tag{57}$$

where

$${}_2F_1(a',b';c';z)=1+\frac{a'b'}{c'}z+\frac{a'(a'+1)b'(b'+1)}{c'(c'+1)}\frac{z^2}{2!}+\frac{a'(a'+1)(a'+2)b'(b'+1)(b'+2)}{c'(c'+1)(c'+2)}\frac{z^3}{3!}+\ldots; \tag{58}$$

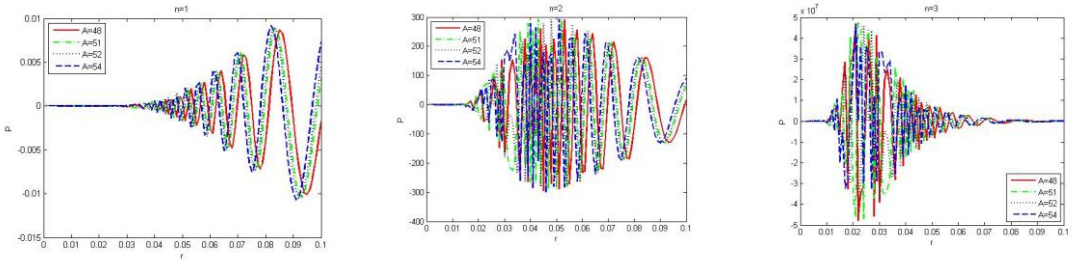
Table 1. The wave function of the Klein-Gordon equation with minimum length formalism for Woods-Saxon potential

n	Wave Function
0	$\psi(0)=z^{\pm C}(1-z)^{\pm\sqrt{(C^2-B^2+A^2)}}$
1	$\psi(1)=z^{\pm C}(1-z)^{\pm\sqrt{(C^2-B^2+A^2)}}\times\left(1+\frac{(-1)(2x+2y+2)}{1+2y}z\right)$
2	$\psi(2)=z^{\pm C}(1-z)^{\pm\sqrt{(C^2-B^2+A^2)}}\times\left(1+\frac{(-2)(2x+2y+3)}{1+2x}z+\frac{(2)(2x+2y+3)(2x+2y+4)}{(1+2x)(2+2x)}\frac{z^2}{3!}\right)$

The quantum number n = 0, 1, 2, 3 for the wave function are served in Table 1. From Table 1, we can see that the unnormalized wave function for ground state indicated by n = 0, energy extent 1 for n = 1, and energy extent 2 for n = 2. The unnormalized wave functions are shown in Figure 1 for different atoms.

The atomic masses selected in this study were 48, 51, 52, and 55 and the energy spectrum shown in Tables 2, 3, and 4. It can be observed that the spectral energy decreases in value as the quantum number (*n*) and angular quantum number (*L)* for different atoms

increase. The increase in atomic mass number also makes the energy value bigger, although it can be seen in the table that the increase is not overly significant.



(a) (b) (c)

Figure 1 . The graph of unnormalized wave function with $L = 1$ for (a) n = 1, (b) n = 2, and (c) n = 3

The energy equation of the Klein-Gordon equation for the Woods-Saxon potential with minimal length formalism is shown by equation (56). The energy result from equation (56) cannot be calculated analytically, so that the energy spectrum is calculated by the Matlab program numerically. The energy spectrum is calculated for variations of n, L, and α_{ML} which are the quantum number, angular momentum number, and minimum length parameter, respectively. The energy spectrum for the Woods-Saxon potential is presented in Table 2, 3, and 4.

Table 2. The energy spectrum with α_{ML} = 0 and n = 1 for various L

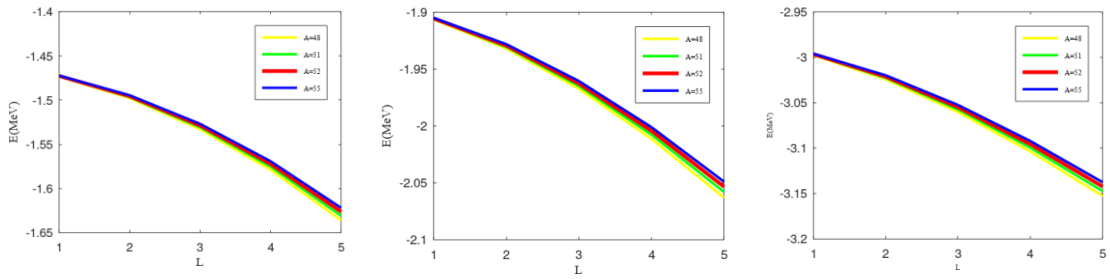
L	E (MeV)			
	A=48	A=51	A=52	A=55
0	-1.460908	-1.460908	-1.460908	-1.460908
1	-1.473512	-1.473084	-1.472691	-1.472326
2	-1.497835	-1.496647	-1.495547	-1.494527
3	-1.533009	-1.530804	-1.528758	-1.526853
4	-1.578817	-1.575353	-1.572132	-1.569127
5	-1.635785	-1.630777	-1.626116	-1.621765

Table 3. The energy spectrum with α_{ML} = 0 and n = 3 for various L

L	E (MeV)			
	A=48	A=51	A=52	A=55
0	-1.892857	-1.892857	-1.892857	-1.892857
1	-1.906399	-1.905891	-1.905423	-1.904992
2	-1.931881	-1.930504	-1.929232	-1.928054
3	-1.967330	-1.964873	-1.962594	-1.960475
4	-2.011289	-2.007616	-2.004198	-2.001010
5	-2.062947	-2.057941	-2.053274	-2.048910

Table 4. The energy spectrum with α_{ML} = 0 and n = 5 for various L

L	E (MeV)			
	A=48	A=51	A=52	A=55
0	-2.983298	-2.983298	-2.983298	-2.983298
1	-2.997685	-2.997082	-2.996529	-2.996021
2	-3.024286	-3.022676	-3.021191	-3.019819
3	-3.060342	-3.057522	-3.054910	-3.052485
4	-3.103650	-3.099534	-3.095709	-3.092145
5	-3.152758	-3.147308	-3.142229	-3.137483



(a) (b) (c)

Figure 2 . The energy graph for Woods-Saxon potential as a L function with $\alpha_{ML} = 0$ for (a) $n = 2$, (b) $n = 3$, and (c) $n = 5$

The results of the energy obtained are negative indicating that this equation is applied to single particles. There is a constant potential difference in single particles. The energy graph in Tables 2, 3, and 4 are presented in Figures 2(a), 2(b), and 2(c). Those figures show the energy spectrum without minimal length effect or for $\alpha_{ML} = 0$, the energy value depends on the radial quantum number (n) and the orbital quantum number (L), we can see that the maximum energy for n = 2, 3, 5 is at lower L.

Table 5. The energy spectrum with $\alpha_{ML} = 0.005$ for various n and L

n	L	E (MeV)			
		A = 48	A = 51	A = 52	A = 55
3	1	-1.520843	-1.521084	-1.521303	-1.521502
	2	-1.523300	-1.523605	-1.523894	-1.524167
	3	-1.552903	-1.549834	-1.547515	-1.545715
4	1	-1.744252	-1.744609	-1.744934	-1.745231
	2	-1.736349	-1.737310	-1.738186	-1.738988
	3	-1.728320	-1.729875	-1.731305	-1.732622
5	1	-2.003358	-2.003710	-2.004029	-2.004321
	2	-1.994037	-1.995037	-1.995947	-1.996779
	3	-1.981795	-1.983628	-1.985303	-1.986836

Table 6. The energy spectrum with $\alpha_{ML} = 0.04$ for various n and L

n	L	E (MeV)			
		A = 48	A = 51	A = 52	A = 55
6	1	-1.994168	-1.994413	-1.994639	-1.787618
	2	-1.788609	-1.788772	-1.981915	-1.982569
	3	-1.958131	-1.792231	-1.961343	-1.792558
	4	-1.801819	-1.801575	-1.801379	-1.932322
7	1	-2.170922	-2.171128	-1.931895	-1.931943
	2	-2.159428	-1.932647	-1.932792	-1.932921
	3	-1.934469	-2.142582	-1.934953	-1.935157
	4	-2.114564	-2.117042	-1.939868	-1.940035
8	1	-2.341966	-2.079134	-2.079181	-2.079224
	2	-2.331932	-2.332484	-2.332992	-2.080085
	3	-2.316244	-2.081472	-2.318450	-2.319427
	4	-2.084725	-2.085015	-2.085274	-2.085505
9	1	-2.228623	-2.507920	-2.508068	-2.508204
	2	-2.498750	-2.499240	-2.499691	-2.500108
	3	-2.484754	-2.230977	-2.486699	-2.231359
	4	-2.465009	-2.466791	-2.468430	-2.234531

Tables 5 and 6 show the energy values with for variations n and L with the formalism of minimal length $\alpha_{ML} = 0.005$ and $\alpha_{ML} = 0.04$. In the table, it can be observed that the

energy value decreases with the increase in the quantum number and angular momentum. The energy value also shows a negative interpretation. The energy value increases slightly as the atomic mass number increases. It can be seen that the Klein-Gordon equation which is affected by the minimum length produces a larger energy value than that which is not affected by a small increase.

4. Conclusion

The equation of Klein-Gordon with minimum length effect for Woods-Saxon has been solved in this paper. Energy spectra and non-normalized wave functions were gotten by implementing the NUFA method. The energy equation that we obtained can be seen in equation (56), while the wave function for variations in quantum numbers *n* can be seen in Table 1. The spectum energy is calculated numerically and the results can be seen in table 2, 3, 4, 5, and 6. The energy obtained is negative because the Klein-Gordon equation is affected by the constant Woods-Saxon potential.

The energy calculation showed that the energy spectrum was influenced by angular momentum, quantum number, minimum length parameter, and number mass of atom. We can see in Table 2, 3, and 4 that the higher the quantum number and angular momentum, the lower the energy. In contrast to the minimum length in Table 5 and 6, it showed that the energy spectrum will increase in value when the minimum length parameter is enlarged. An increase in atomic mass also causes energy to increase as the radial quantum number and angular momentum are held constant.

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