Analytical investigation of spin-orbit interaction effect on Hulthen potential model using perturbation theory

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Abstract: This study investigates the effect of Spin-Orbit interaction on Hulthen potential model within the non-relativistic quantum mechanical domain. The Schrodinger equation is solved for a particle with spin degree of freedom moving in a central force field modelled by the Hulthen potential. The Spin-Orbit interaction is viewed as a form of perturbation. Hence the perturbation theory is applied to correct the energy eigenvalues of the Hulthen system, thereby revealing the effect of Spin-Orbit effect. The results of this study shows that the energy levels of the Hulthen potential splits according to the $j = \ell \pm {}^{1}/_{2}$ scheme except for $\ell = 0$. This makes the Hulthen potential model suitable for a system with spin degree of freedom within the non-relativistic quantum mechanics. Also, Schrodinger equation which is apt for a spinless particle can be applied to study a system with spin degree of freedom by treating the Spin-Orbit interaction as a form of perturbation.

Keyword: Schrodinger equation, Hulthen potential, perturbation theory, Spin-orbit interaction, energy eigenvalues

1. Introduction

Solving the Schrodinger wave equation for various potential models to obtain the energy eigenvalue and the corresponding wave function has been an exciting area of research in physics and chemistry in recent times. The Schrodinger wave equation has been solved by quite a number of researchers using various potential models. Commonly applied potential models applied to solve the Schrodinger equation includes; the Coulomb potential [1, 2], Eckart potential [3, 4], Hulthen potential [5, 6], Hylleraas potential [7, 8], Woods – Saxon potential [9 -10], Yukawa potential [11. 12], and in some cases the potential combined to construct a more effective potential model and used to this equation. Recently, some combined potentials used by scholars to solve the Schrodinger equation are; SHSC potential [13], Hulthen – Hellmann potential [14], Hulthen – Coulomb potential [15], Kratzer – Yukawa potential [16], Varshni – Kratzer potential [17], Manning - Rosen – Inversely quadratic Yukawa potential [18], Hua – Modified Eckart potential [19], Hulthen – Ring – shaped potential [20], and many others found in several literatures.

However, real physical systems are often times subjected to both external and internal forces which may not be represented effectively by a single potential model mostly used in describing such systems within the domain of quantum mechanics. As such the energy eigenvalues for a given potential model may not give the correct energy structure when such is used to model a real physical system. The essence of using potential model in quantum mechanical studies is to replicate real physical situation as much as possible. But most potential models used in quantum mechanical studies of physical systems are usually limited to modelling the central force prevalent in such systems, and often times other constraints, such as Spin-Orbit interactions, magnetic effects, etc are ignored.

To solve this problem the perturbation theory becomes essential, as it allow other interactions or constraints within the system under study which are not represented by the potential model used to model the central force to be factored into the problem with minimal mathematical effort.

There are various methods of solving the Schrodinger wave equation. Common amongst the methods are the asymptotic iteration method [21], super-symmetric shape invariance approach [22], exact or proper quantization rule [23], factorization method [24], 1/N shifted expansion method [25] and Nikiforov - Uvarov method [26].

The aim of this study is to investigate the Spin-Orbit effect on the Hulthen potential using the perturbation theory. The Hulthen potential is a central mean field potential that represents a short-range force field. In this research, the Hulthen potential is viewed as representing a strong force field, which influences nucleon interactions within the nucleus or any quarkunian system of any configuration. This implies that the Hulthen potential could be applied to study the nucleon-nucleon (N-N) or any particle moving in a strong force field.

The spin-orbit interaction is an important component of the force field under which a particle with spin degree of freedom moves [26]. The spin-orbit effect occurs when a particles possessing the spin degree of freedom and magnetic moment interacts with the mean field potential. This effect is usually visible for orbits with $\ell \geq 0$ [27]. Hence, recognizing the spin - orbit effect and giving it a consideration when studying the motion of a particle with spin in the mean field potential is of essence. The spin-orbit effect for a spin ½ particle is easily studies within the relativistic quantum mechanical domain, where the spin is effectively represented in the wave equation (Dirac equation) [28]. Although no consideration is given to the spin of a particle in the non-relativistic Schrodinger wave equation. However, in this study the spin has been introduced to a non-relativistic quantum mechanical study by treating the spin-orbit interaction with the Hulthen potential as a form of perturbation.

The statement of problem is for this study is clearly stated as follows, Spin–orbit contributions to the energy spectrum of Hulthen system within the non-relativistic quantum mechanics is often neglected. Consequently, there is limited analytical understanding of how spin–orbit coupling perturbs the bound-state energy levels and wavefunctions under the Hulthen potential. This research seeks to fill this gap by employing perturbation theory to derive analytical expressions for the energy corrections and thus elucidating the effect of spin–orbit interaction on quantum systems governed by the Hulthen potential. The Spin-orbit interaction term is not an integral part of the Hamiltonian of the Hulthen system as such it can be treated as a form of perturbation.

The spin-orbit interaction term is given as $V_{LS}(r)=\frac{1}{2}V_{LS}(0)\left(\frac{r_o}{\hbar}\right)^2\frac{1}{r}\left[\frac{dV_H(r)}{dr}\right]\vec{L}.\vec{S}$, where $\vec{L}.\vec{S}$ is the spin-orbit coupling,

In this study, the effect of the spin-orbit interaction on Hulthen potential is investigated. The aim of this study is achieved by solving analytically the Schrodinger wave equation for a spin ½ particle moving in a central field represented by the Hulthen potential. The Schrodinger equation is solved using the Nikiforov-Uvarov (NU) method. The spin-orbit interaction due to the motion of the particle is treated as a form of perturbation with the Hulthen system, hence perturbation theory is applied to obtain the first order correction to energy of the system. Furthermore, the Hellman-Feynman theorem is used to obtain the expectation value of the radius of the system. This is required to determine analytically the spin-orbit interaction term.

2. Methodology

We take a brief overview of the NU method. The details is as presented by Nikiforov and Uvarov [29] This method involves solving a hyper-geometric type second order differential equation:

$$\psi^{||}(z)+\frac{\tilde{\tau}(z)}{\sigma(z)}\psi^{||}(z)+\frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi(z)=0 \tag{1}$$

Where $\sigma(z)$ and $\tilde{\sigma}(z)$ are second order polynomials. $\tilde{\tau}(z)$ is a first order degree polynomial. [30]. A possible solution to Eq. (1) is proposed as:

$$\psi(z)=\phi(z)y(z) \tag{2}$$

Which results in a hyper geometric – type equation of the form;

$$\sigma(z)y^{||}(z)+\tau(z)y^{|}(z)+\lambda y(z)=0 \tag{3}$$

Part of the solution to (1) given in (2) as $\phi(z)$ is the solution to another differential equation of the form given by

$$\sigma(z)\phi^{|}(z)-\pi(z)\phi(z)=0 \tag{4}$$

Where

$$\tau(z)=\breve{\tau}(z)+2\pi(z) \tag{5}$$

And λ in (3) is defined as

$$\lambda=\lambda_n=-n\tau^{|}(z)+\frac{n(n-1)}{2}\sigma^{||}(z)=0(n=0,1,2,3,...) \tag{6}$$

The term $\tau(z)$ is a polynomial, and to obtain a proper solution for the hyper-geometric type differential equation, its first derivative $\tau^{|}(z)$ must be negative. The function $y(z)$ as stated in (2) is the hyper geometric – type wave function obtained using the Rodrigues relation:

$$y_n(z)=\frac{B_n(z)d^n}{\rho(z)dz^n}[\sigma^n(z)\rho(z)] \tag{7}$$

Where B_n is related to the normalization constant, and $\rho(z)$ is defined as

$$\frac{d}{dz}[\sigma(z)\rho(z)] = \tau(z)\rho(z) \tag{8}$$

Also the function $\pi(z)$ is a first order polynomial is defined as

$$\pi(z) = \frac{\sigma^{|}(z) - \tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma^{|}(z) - \tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z)} \tag{9}$$

K in (9) is related to (6) and the first derivative of $\pi^{|}(z)$ as thus

$$\lambda = k + \pi^{|}(z) \tag{10}$$

The value of k is obtained by equating the discriminant of the quadratic expression under the square root sign in (10) to zero. By solving (6) and (10), we derive the energy eigenvalue equation.

Time independent perturbation theory is an approximate scheme that applies to situations where the solution to the eigenvalue problem of the Hamiltonian H^0 is known. But a solution to the Hamiltonian of the form given by Eq. (11) is unknown.

$$H = \lambda^0 H^0 + \lambda^1 H^1 \tag{11}$$

Assuming for every eigenket $\left|E_n^0\right\rangle \equiv \left|n^0\right\rangle$ of H^0 with eigenvalue E_n^0 there is an eigenket $\left|n\right\rangle$ of H with eigenvalue E_n. Expanding the eigenkets and the eigenvalues of H in perturbation series as shown in (12) and (13) respectively.

$$\left|n\right\rangle = \lambda^0\left|n^0\right\rangle + \lambda^1\left|n^1\right\rangle + \lambda^2\left|n^2\right\rangle + \ldots \tag{12}$$

$$\left|E_n\right\rangle = \lambda^0\left|E_n^0\right\rangle + \lambda^1\left|E_n^1\right\rangle + \lambda^2\left|E_n^2\right\rangle + \ldots \tag{13}$$

The superscripts in (12) and (13) defines the order of perturbation. Where λ is a small parameter and a constant usually considered to be less than 1, but in perturbation theory it is used to keep track the order of the magnitude of different terms of the expansion that underlies the theory [31].

Given the Hamiltonian H as defined by (11), the eigenvalue equation (14) is rewritten by substituting (12) and (13) into (14) to obtain (15).

$$H\left|n\right\rangle = E_n\left|n\right\rangle \tag{14}$$

$$\left(\lambda^0 H^0 + \lambda^1 H^1\right)\left[\lambda^0\left|n^0\right\rangle + \lambda^1\left|n^1\right\rangle + \ldots\right] = \left(\lambda^0 E_n^0 + \lambda^1 E_n^1\right)\left[\lambda^0\left|n^0\right\rangle + \lambda^1\left|n^1\right\rangle + \ldots\right] \tag{15}$$

The zeroth order terms of (15) is given as (16). This is obtained by considering fact that terms are proportional to λ. However, it should be noted that the small parameter λ in itself has no physical meaning but only useful in the perturbation series expansion to track the order of perturbation.

$$H^0\left|n^0\right\rangle = E_n^0\left|n^0\right\rangle \tag{16}$$

Also, the first order term of (15) is stated as (17).

$$H^0\left|n^1\right\rangle + H^1\left|n^0\right\rangle = E_n^0\left|n^1\right\rangle + E_n^1\left|n^0\right\rangle \tag{17}$$

We dot both sides of (17) with $\left\langle n^0\right|$ using $\left\langle n^0\right|H^0 = \left\langle n^0\right|E_n^0$ and $\left\langle n^0 | n^0\right\rangle = 1$ to obtain the perturbed energy given as (18).

$$E_n^1 = \left\langle n^0\right|H^1\left|n^0\right\rangle \tag{18}$$

Dot both side of (17) with $\left\langle m^0\right|$ given that $m \neq n$ to obtain the perturbed eigenket as given as (19).

$$\left|n\right\rangle = \left|n^0\right\rangle + \sum_m^1 \frac{\left|m^0\right\rangle\left\langle m^0\right|H\left|n^0\right\rangle}{E_n^0 - E_m^0} \tag{19}$$

The `Schrodinger wave equation is now solved for the Hulthen potential model to obtain the eigenvalues.
The Schrodinger equation is given as

$$\frac{d^2R(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[E_{n\ell j} - V_{eff}(r)\right]R(r) = 0 \tag{20}$$

Where, μ is the reduced mass, $\hbar$ is the Planck's constant, V_{eff} is the effective potential, $E_{n\ell j}$ is the energy spectrum, and n, ℓ and j are the principal, orbital and total momentum quantum number respectively. We begin by transforming the radial function R(r) of the Schrodinger wave equation by introducing a new variable defined as

$$z = e^{-2\alpha r} \tag{21}$$

Taking the second derivative of (21) with respect to r, and redefining R(r) as $\psi(z)$, (20) is transformed as thus;

$$\frac{d^2\Psi(z)}{d^2Z} + \frac{1}{Z}\frac{d\Psi(z)}{dZ} + \frac{1}{\alpha^2 Z^2}\left[\frac{2\mu E}{\hbar^2} - \frac{2\mu}{\hbar^2}V_{eff}\right]\Psi(z) = 0 \tag{22}$$

The effective potential V_{eff} is given by (23).

$$V_{eff} = \frac{-V_0\alpha e^{-\alpha r}}{1-e^{-\alpha r}} + \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2} \tag{23}$$

Where α is the screening parameter, V_o is the potential strength of the Hulthen potential, $\frac{\ell(\ell+1)}{r^2}$ is the centrifugal term and μ is the reduced mass.

We applying the Greener – Aldrich approximation which is of the form given by (24) and (25) to handle the centrifugal term.

$$\frac{1}{r} = \frac{\alpha e^{-\alpha r}}{\left(1-e^{-\alpha r}\right)} \tag{24}$$

$$\frac{1}{r^2} = \frac{\alpha^2 e^{-2\alpha r}}{\left(1-e^{-\alpha r}\right)^2} \tag{25}$$

The Schrodinger wave equation is now written in form of hyper- geometric second order differential equation given by (26).

$$\frac{d^2\Psi(z)}{d^2Z}+\frac{(1-Z)}{Z(1-Z)}\frac{d\Psi(z)}{dZ}+\frac{1}{Z^2(1-Z)^2}\begin{bmatrix}(-\gamma_1-\gamma_2)Z^2+\\(\gamma_2+2\gamma_1+\gamma_3)Z\\-\gamma_3\end{bmatrix}\Psi(z)=0 \tag{26}$$

Where; $\gamma_1=\frac{2\mu E}{\alpha^2\hbar^2}$ $\gamma_2=\frac{2V_0\mu}{\alpha\hbar^2}$ $\gamma_3=\ell(\ell+1)$

Comparing (26) to (11) the following polynomials are defined as stated in equations (27), (28), (29) and (30)

$$\tilde{\tau}(z)=1-Z \tag{27}$$

$$\sigma(z)=Z(1-Z) \tag{28}$$

$$\sigma^2(z)=Z^2(1-Z)^2 \tag{29}$$

$$\tilde{\sigma}(z)=(-\gamma_1-\gamma_2)Z^2+(\gamma_2+2\gamma_1-\gamma_3)Z-\gamma_3 \tag{30}$$

Let;

$$\beta_1=\frac{1}{4}+\gamma_1+\gamma_2 \qquad \beta_2=-\gamma_2-2\gamma_1+\gamma_3 \qquad \beta_3=\gamma_3$$

Putting (27), (28), (29) and (30) into (9), we have

$$\pi(z)=\frac{-Z}{2}\pm\sqrt{(\beta_1-k)Z^2+(\beta_2+k)Z+\beta_3} \tag{31}$$

Equating the discriminant of the quadratic under the square root sign in (31) to zero and solving the resulting equation for k and substituting it back into (31). We is obtained

$$\pi(z)=\frac{-Z}{2}\pm\left(\sqrt{\beta_3}+\sqrt{\beta_1+\beta_2+\beta_3}\right)Z-\sqrt{\beta_3} \tag{32}$$

Taking the first derivative of (32), we have

$$\pi^{|}(z)=\frac{-1}{2}\pm\left(\sqrt{\beta_3}+\sqrt{\beta_1+\beta_2+\beta_3}\right) \tag{33}$$

Putting k obtained from (31), and (33) into (10), we have,

$$\lambda=\frac{-1}{2}-\sqrt{\beta_3}-\sqrt{\beta_1+\beta_2+\beta_3}-\beta_2-2\beta_3-2\sqrt{\beta_3}\sqrt{\beta_1+\beta_2+\beta_3} \tag{34}$$

Substituting (27) and (32) into (5), we have

$$\tau(z)=1-2Z-2\left(\sqrt{\beta_3}+\sqrt{\beta_1+\beta_2+\beta_3}\right)Z-2\sqrt{\beta_2} \tag{35}$$

Taking the first derivative of (35), we have

$$\tau^{|}(z)=-2-2\left(\sqrt{\beta_3}+\sqrt{\beta_1+\beta_2+\beta_3}\right) \tag{36}$$

Substituting (36) into (6), we have

$$\lambda_n=n^2+n+2n\sqrt{\beta_3}+2n\sqrt{\beta_1+\beta_2+\beta_3} \tag{37}$$

Equating (34) and (37) and factorizing some terms. The energy eigenvalues $E_{n\ell}$ for the Hulthen potential is obtained as;

$$E_{n\ell} = \frac{\alpha^2\hbar^2}{8\mu}\left[\frac{\left(n+\ell+1\right)^2 - \frac{2V_0\mu}{\alpha\hbar^2}}{\left(n+\ell+1\right)}\right]^2 \tag{38}$$

We obtain normalized wave function as;

$$\Psi(z) = N_n Z^{\frac{\sqrt{2\mu E}}{\alpha^2\hbar^2}}(1-Z)^{\frac{1}{2}-\frac{(2\ell+1)}{2}} P_n^{\left[2\frac{\sqrt{2\mu E}}{\alpha^2\hbar^2},2\frac{(2\ell+1)}{2}\right]}(1-2Z) \tag{39}$$

The spin-orbit interaction [15] is as given in (40)

$$V_{LS}\left(r\right) = \frac{1}{2}V_{LS}(0)\left(\frac{r_o}{\hbar}\right)^2\frac{1}{r}\left[\frac{dV_H(r)}{dr}\right]\vec{L}.\vec{S} \tag{40}$$

Where the spin-orbit coupling $\vec{L}.\vec{S} = \frac{\hbar^2}{2}\left(j(j+1)-\ell(\ell+1)-\frac{3}{4}\right)$, V_H is the Hulthen potential function, $V_{LS}(0)\propto V_0$, V_0 is the Hulthen potential strength, r_o is adjustable parameter.

We apply the Hellmann-Feynman theorem [32] as stated in (41) to the energy spectrum of the Hulthen system given in (38), the expectation value for position of the particle is obtained. The reciprocal of position vector r which appears in the expression of the spin-orbit interaction is determined, as given by (42).

$$\left\langle \psi_q \middle| \frac{\partial H}{\partial \ell} \middle| \psi_q \right\rangle = \frac{\partial E}{\partial \ell} \tag{41}$$

$$\frac{1}{r} = \frac{\alpha}{2}\left\{\frac{\left[\frac{\left(n+\ell+1\right)\left[\frac{4\left(n+1\right)}{2\ell+1}\right]-\left[\left(n+\ell+1\right)^2-\frac{2V_0\mu}{\hbar^2\alpha}\right]\left(\frac{2}{2\ell+1}\right)}{\left(n+\ell+1\right)^2}\right]}{\left[\frac{\left(n+\ell+1\right)^2-\frac{2V_0\mu}{\hbar^2\alpha}}{\left(n+\ell+1\right)}\right]}\right\}^{\frac{1}{2}} \tag{42}$$

Putting (42) into (40), we have the spin-orbit interaction

$$V_{LS} = \left[-0.11 V_0^3 \alpha^4 r_o^2 \left\{ \frac{\left[\left(n+\ell+1\right)\left[\frac{4\left(n+1\right)}{2\ell+1}\right] - \frac{\left[\left(n+\ell+1\right)^2 - \frac{2V_0\mu}{\hbar^2\alpha}\right]\left(\frac{2}{2\ell+1}\right)}{\left(n+\ell+1\right)^2} \right]}{\left[\frac{\left(n+\ell+1\right)^2 - \frac{2V_0\mu}{\hbar^2\alpha}}{\left(n+\ell+1\right)}\right]} \right\}^{1/2} \left(j(j+1) - \ell(\ell+1) - \frac{3}{4} \right) \right] \tag{43}$$

Where $j = \ell \pm \frac{1}{2}$

Therefore, we derive the correction to the energy eigenvalues E of the Hulthen potential system as a result of the spin-orbit interaction with the Hulthen potential. According to perturbation theory, the correction to energy of the Hulthen system is stated explicitly by (44).

$$E = E^0 + E^1 \tag{44}$$

E^0 is the unperturbed energy eigenvalues of the Hulthen potential given in (42) while $E^1 = V_{LS}$ is the perturbation due to the spin-orbit interaction which must be added to the unperturbed energy to correct it.

Putting (38) and (43) into (44), we have the corrected energy spectrum of the system considering a spin-orbit interaction with the Hulthen potential as given in (45).

$$E = \frac{\alpha^2\hbar^2}{8\mu}\left[\frac{\left(n+\ell+1\right)^2 - \frac{2V_0\mu}{\alpha\hbar^2}}{\left(n+\ell+1\right)}\right]^2 - \left[0.11 V_0^3 \alpha^4 r_o^2 \left\{ \frac{\left[\left(n+\ell+1\right)\left[\frac{4\left(n+1\right)}{2\ell+1}\right] - \frac{\left[\left(n+\ell+1\right)^2 - \frac{2V_0\mu}{\hbar^2\alpha}\right]\left(\frac{2}{2\ell+1}\right)}{\left(n+\ell+1\right)^2} \right]}{\left[\frac{\left(n+\ell+1\right)^2 - \frac{2V_0\mu}{\hbar^2\alpha}}{\left(n+\ell+1\right)}\right]} \right\}^{1/2} \left(j(j+1) - \ell(\ell+1) - \frac{3}{4} \right) \right] \tag{45}$$

The following are used to calculate the numerical values of the energy eigenvalues of (45) as presented on Table 1; $V_0 = 1, \alpha = 0.5, \hbar = 1, \mu = 1$ and $r = 0.03$.

Table 1. Energy eigenvalues $E_{n\ell j}$ (eV) showing spin-orbit interaction with Hulthen potential

n	ℓ	J	$E_{n\ell j}$
1	0	1/2	-0.101250000000
2	0	1/2	-0.033368055000
	1	1/2	-0.011250004000
		3/2	-0.011249998000
3	0	1/2	-0.0112500000000
	1	1/2	-0.0028125062450
		3/2	-0.0028124967410
	2	3/2	-0.0001388970163
		5/2	-0.0001388347060
4	0	1/2	-0.0028125000000
	1	1/2	-0.0001388967107
		3/2	-0.0001388849780
	2	3/2	-0.0005165914384
		5/2	-0.0005165750934
	3	5/2	-0.0028125117490
		7/2	-0.0028124911880
5	0	1/2	-0.0001388888891
	1	1/2	-0.0005161590880
		3/2	-0.0005165770088
	2	3/2	-0.0028125114350
		5/2	-0.002812492377
	3	5/2	-0.006485353107
		7/2	-0.004853283905
	4	7/2	-0.011250015730
		9/2	-0.011249937410

3. Results

The unperturbed $E_{n\ell}$ energy spectrum of the Hulthen system is given as

$$E_{n\ell} = \frac{\alpha^2\hbar^2}{8\mu}\left[\frac{(n+\ell+1)^2 - \frac{2V_0\mu}{\alpha\hbar^2}}{(n+\ell+1)}\right]^2$$

The spin-orbit interaction E^1 which represent the perturbation due to spin-orbit coupling is obtained in this study, and is given explicitly as

$$E^1 = \left[-0.11 V_0^3 \alpha^4 r_o^2 \left\{ \frac{\left[(n+\ell+1)\left[\frac{4(n+1)}{2\ell+1}\right] - \frac{\left[(n+\ell+1)^2 - \frac{2V_0\mu}{\hbar^2\alpha}\right]\left(\frac{2}{2\ell+1}\right)}{(n+\ell+1)^2} \right]}{\left[\frac{(n+\ell+1)^2 - \frac{2V_0\mu}{\hbar^2\alpha}}{(n+\ell+1)} \right]} \right\}^{1/2} \left(j(j+1) - \ell(\ell+1) - \frac{3}{4} \right) \right]$$

We obtained the correction to energy E of the Schrodinger equation for the Hulthen system taken into consideration the spin-orbit effect on the system as thus;

$$E = \frac{\alpha^2\hbar^2}{8\mu}\left[\frac{(n+\ell+1)^2 - \frac{2V_0\mu}{\alpha\hbar^2}}{(n+\ell+1)} \right]^2 - \left[0.11 V_0^3 \alpha^4 r_o^2 \left\{ \frac{\left[(n+\ell+1)\left[\frac{4(n+1)}{2\ell+1}\right] - \frac{\left[(n+\ell+1)^2 - \frac{2V_0\mu}{\hbar^2\alpha}\right]\left(\frac{2}{2\ell+1}\right)}{(n+\ell+1)^2} \right]}{\left[\frac{(n+\ell+1)^2 - \frac{2V_0\mu}{\hbar^2\alpha}}{(n+\ell+1)} \right]} \right\}^{1/2} \left(j(j+1) - \ell(\ell+1) - \frac{3}{4} \right) \right]$$

The result of this study shows fine structure splitting of the energy levels of the Hulthen potential as presented on Table 1. All the ℓ levels except $\ell = 0$ level are split into two sub levels following the $j = \ell \pm 1/2$ scheme. The splitting of the energy levels is due to the interaction of the spin and the orbital angular momentum. This behaviour is peculiar to any system with spin degree of freedom. However, the Schrodinger wave equation is adapted for a spinless particle, as no consideration is given to the spin of the particle moving within a given potential. But in this study we have been able to factor spin degree of freedom into the solution of the Schrodinger wave equation for the Hulthen potential model by treating the spin-orbit interaction as a form of perturbation.

4. Conclusion

This study shows that the Schrodinger wave equation can be applied to problems on systems with the spin degree of freedom by treating the Spin-Orbit interaction which arose from the spin degree of freedom as a form of perturbation. Also, this study shows that the Hulthen potential model can be applied to a system with spin degree of freedom within the scope of non-relativistic quantum mechanics by simply treating the spin-orbit interaction within the system as a form perturbation.

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Conflict of Interest

The authors declare that they have no conflicts of interest.

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