Modified time-dependent linear thermal expansion equation: using Inhomogeneous 1-D heat equation

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Abstract: The correction factor must be derived from the results of the linear thermal expansion experiment. We have two ways to address this problem: we use the form of polynomials for the linear thermal coefficient, and one must solve the one-dimensional heat diffusion equation. The temperature function that we obtained is the solution for the inhomogeneous differential equation. Using those two, then combine them into a modified linear thermal expansion equation, i.e., the infinitesimal form of the equation, so that we could find the expression for the time-dependent expansion for the metal rod, $\Delta L(t)$. We should attempt to reduce the higher-order terms by taking the approximation as our first step in this paper. Finally, the observer may choose a suitable boundary condition for the formula and use the resulting equation as the correction factor.

Keywords: linear expansion coefficient, heat diffusion, inhomogeneous boundary condition, Fourier series solution

1. Introduction

The thermal expansion experiment is one of the modules frequently conducted in Basic Physics laboratories to observe the physical properties of a one-dimensional metal rod. The physical property studied is the ability of an object to expand when heated. This property is determined by the linear thermal expansion coefficient (LTEC). This experiment can be performed with a simple apparatus, which only requires a metal rod, a heater, and a set of sensitive length measurement tools. Several applications of the linear expansion set and its modifications to support technological advancements in teaching materials have been previously carried out, such as utilizing the Internet of Things as a basis for linear expansion experiments (Santoso et al., 2023) and thermal expansion of solids (Drebushchak, 2020; Zorzi & Perottoni, 2021). Means that the theoretical basis for determining the coefficient is always assumed to be a constant value for the coefficient of expansion. A model that uses a polynomial function for the coefficient of linear expansion has already been done by other studies. As a result, it is necessary to examine how the elongation of a heated metal rod depends on time. The previous argument is reinforced by the following experimental set.

Figure 1. The experiment set that utilized for the addressed linear thermal expansion problem.

From the figure above, linear expansion happened when the hot water steam flowed through the tube from the right end to the left end, then condensed. At the middle point of the tube, we can see the thermometer, which is used to measure the supposed average temperature inside the tube. Here, we have, for example, some data results obtained from the apparatus.

Table 1. Average temperature achieved from 10 experiments of thermal expansion for an iron rod, with six different elongations.

No.	ΔL (m)	Temperature(°C)
1	0	25
2	0.00005	26
3	0.0001	27
4	0.00015	27
5	0.00002	27
6	0.00025	63

Using the constant thermal expansion coefficient model, we can find the error ranged from 7 to 20 %. Therefore, a modification of the theory regarding linear thermal expansion is needed to find its time dependency. This study will involve using the heat diffusion equation to produce the temperature dynamics solution along the rod, and also the polynomial model of the LTEC to enhance the modification. The expression obtained will be used in the expansion equation, and one can get the equation for time-dependent elongation for the metal rod.

2. Analysis Setup

2.1. *Determining generalized linear thermal expansion*

A heated 1-D metal rod satisfies the following infinitesimal equation (Drebushchak, 2020)

$$\alpha = \frac{1}{L_0}\frac{dx}{dT}, \tag{1}$$

with α, L_0, and $\frac{dx}{dT}$ are the linear thermal coefficient, the initial rod length, and the rate of change of length over temperature. If we integrate equation (1), we can get

$$\int dx = \int \alpha L_0 dT \tag{2}$$

$$\delta L = \alpha(T)\delta x \Delta T. \tag{3}$$

The integration of equation (2) into equation (3) makes it clear that we take into account the coefficient of linear expansion as a function of time, $\alpha(T)$, and the initial length L_0 as a temperature-dependent partition of the rod's length, δx. In the meantime, the division of length expansion caused by a temperature change, ΔT, is represented by δL.

2.2. *Determining 1-D time-dependent linear thermal expansion*

To generate a suitable expression for the modified linear thermal expansion, we can look at the equipment setup below

Figure 2. The hot steam enters from a single point on the right end, whereas condensed at the opposite end.

According to the equipment above, means that the following equation can be formulated as a fundamental model for the modified theory

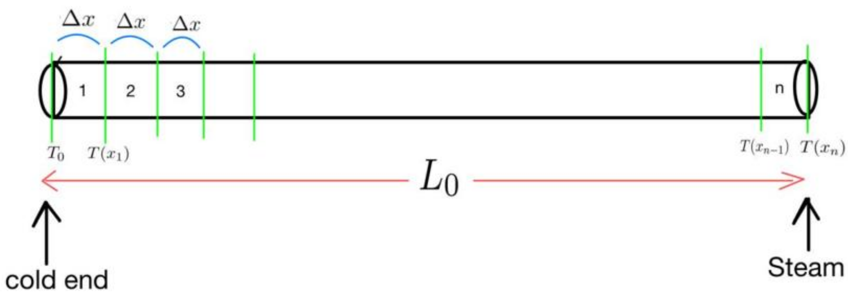

Figure 3. Length element partition of a heated metal rod.

From the diagram above, it can be seen that the rod's length is divided into n elements. For each element can be expressed as follows

$$\begin{aligned}\delta L_1 &= \alpha(T)\Delta T\delta x_1,\\ \delta L_2 &= \alpha(T)\Delta T\delta x_2,\\ \delta L_3 &= \alpha(T)\Delta T\delta x_3,\\ &\vdots\\ \delta L_n &= \alpha(T)\Delta T\delta x_n.\end{aligned} \quad (4)$$

Each element undergoes a change of temperature $\Delta T = T(x+\delta x, t) - T(x,t)$, which can be interpreted as the temperature difference between element n and $(n+1)$ by a position-dependent heat source $Q(x)$. If the maximum partition is set to be $n = N$, then equation (4) can be written as

$$\sum_{n=1}^{N} \delta L_n = \sum_{n=1}^{N} \alpha(T)\Delta T\delta x_n\,. \quad (5)$$

For $N \to \infty$, one can get the equation

$$\lim_{N\to\infty}\sum_{n=1}^{N} \delta L_n = \lim_{N\to\infty}\sum_{n=1}^{N} \alpha(T)\Delta T\delta x_n\,. \quad (6)$$

The equation above can be treated as an integral form as

$$\Delta L = \int_0^L dL = \int_0^L \alpha(T)(T(x,t) - T_0)dx. \quad (7)$$

The equation (7) is used to calculate the total elongation, which can be integrated along the rod as long as the expression for temperature-dependent LTEC is obtained. We also must have the form of the temperature function that depends on the position $T(x)$ where the initial temperature T_0 is to be given.

2.3. *Choosing the generalized LTEC*

This study chooses the most general form for the LTEC, which is the power series as (Kostanovskiy et al., 2022)

$$\alpha(T) = \sum_{p=0}^{\infty} \alpha_p T^p = \alpha_0 + \alpha_1 T + \alpha_2 T^2 + \cdots. \quad (8)$$

3. Solution of Position-dependent Temperature Function

The temperature function for equations (7) and (8) must be obtained from solving the differential equation of heat diffusion. We start by looking at the homogeneous boundary condition problem, which is a prerequisite for solving the inhomogeneous problem.

3.1. *Diffusion equation: homogeneous boundary condition*

We consider a one-dimensional rod of total length L, where the temperature distribution along the rod satisfies the following equation (Kolokolov, 2025)

$$\dot{T} = kT''. \quad (9)$$

For the notation we used $\dot{T} = \partial T/\partial t$ as the change of temperature over time, $T' = \partial T/\partial x$ as the changing temperature over position along the rod, and k is the thermal diffusivity constant for the material. So, the euqtion above must be valid for $0 \leq x \leq L$ and $t > 0$. The initial condition is given by

$$T(x, 0) = f(x), \tag{10}$$

with $f(x)$ is a known function describing the initial temperature distribution profile along the rod. For homogeneous boundary conditions, we take the simplest form as

$$T(0, t) = T(L, t) = 0, t > 0. \tag{11}$$

An example of equation (11) is when a linear rod is heated, where both ends are connected to an extremely good heat sink. Solving the equation above, one can take the form of the solution of the temperature as

$$T(x, t) = \chi(x)\theta(t), \tag{12}$$

where $\chi(x)$ and $\theta(t)$ are the position and time-dependent functions. Subtituting equation (12) into (11) then we can get

$$\frac{\chi''}{\chi} = \frac{1}{k}\frac{\dot{\theta}}{\theta} = -\lambda. \tag{13}$$

Here, we introduce λ as a separation constant. This leads to two differential equaitions

$$\chi'' + \lambda\chi = 0; \quad \dot{\theta} + \lambda k\theta = 0, \tag{14}$$

We use $\chi(0) = \chi(L) = 0$, so these conditions must be an eigenvalue problem. The general solution is

$$\chi_n(x) = \mathrm{A} \sin ux + B \cos ux\,. \tag{15}$$

With some proper boundary conditions, we can get the position function as

$$\chi_n(x) = \sin\left(\frac{n\pi x}{L}\right). \tag{16}$$

Also, for the temporal function, we have

$$\theta_n(t) = \exp\left(-\frac{n^2\pi^2 kt}{L^2}\right). \tag{17}$$

Therefore, the complete solution for the temperture is

$$T_n(x, t) = \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2 k}{L^2}t}, \tag{18}$$

And finally, we have the general solution to the homogeneous linear differential equation, which is expressed as a Fourier series

$$T(x, t) = \sum_{n=1}^{\infty} b_n T_n(x, t)\,, \tag{19}$$

where Fourier coefficients given by $b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx$.

3.2. *Diffusion equation: inhomogeneous boundary condition*

In this study, we examine the following form of the differential equation as a first step toward developing a more general class of boundary conditions as (Mustafa A. Sabri, 2022)

$$\dot{T} - kT'' = Q(t), \tag{20}$$

by boundary conditions where the temperature at one end is held constant $T(0, t) = T_1$, while the temperature at the other end varies with time $T(L, t) = g(t)$, and the initial profile for temperature distribution as $T(x,0) = f(x)$. We already knew that, the general form of the temperature solution for equation (20) has the expression (Kolokolov, 2025)

$$T(x,t) = P(x,t) + R(x,t), \tag{21}$$

where

$$R(x,t) = T_1 + \frac{g(t) - T_1}{L}x. \tag{22}$$

The function $P(x,t)$ must satisfy the homogeneous heat diffusion equation as in equation (19).

4. Results

By substituting equation (8) into (7), the total thermal expansion of the rod can be expressed as

$$\Delta L(t) = \int_0^L (\alpha_0 + \alpha_1 T + \alpha_2 T^2 + \cdots)(T - T_0)dx. \tag{23}$$

For simplicity, the expression before can be rewritten as

$$\Delta L = \int_0^L \sum_{p=0}^{\infty} \alpha_p T^{p+1}\, dx - T_0 \int_0^L \sum_{p=0}^{\infty} \alpha_p T^p\, dx\,. \tag{24}$$

To illustrate a simplified case, let us take $T_0 = T_1 = k = L = 1,\ g(t) = t$, and the initial profile we choose can be represented by $f(x) = 1 + x$ which corresponds to following graph

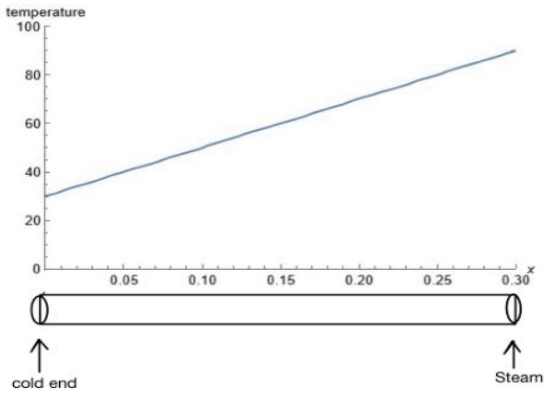

Figure 4. A rod initially 30°C at the cold end (room temperature thermal quilibrium), with steam at 90°C induced at the hot end when $t = 0$.

As a result, we have

$$R(x,t) = 1 + (t - 1)x. \tag{25}$$

Then, as for the function $P(x)$ must satisfies the homogeneous heat diffusion equation and is given by

$$P(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x\, e^{-n^2\pi^2 t}\,, \tag{26}$$

Where the Fourier coefficient

$$b_n = 2\int_0^1 (1+x)\sin n\pi x = 2\left(\frac{1-2(-1)^n}{n\pi}\right). \tag{27}$$

Thus, we have

$$P(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-2(-1)^n}{n}\right)\sin n\pi x\, e^{-n^2\pi^2 t}. \tag{28}$$

Combining both components, the complete form for the temperature solution is

$$T(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-2(-1)^n}{n}\right)\sin n\pi x\, e^{-n^2\pi^2 t} + 1 + (t-1)x. \tag{29}$$

The equation (24) tells us the general formulation for total linear thermal expansion, assuming LTEC α_p and temperature fuction $T(x,t)$ are known. However, to simplify the temperature function we have from equation (29), one can shorten the power series into two terms only by taking the assumption $\alpha_p \approx 0$ for $p \geq 2$. This yields

$$\Delta L = \int_0^1 (\alpha_0 T + \alpha_1 T^2)dx - \int_0^L (\alpha_0 + \alpha_1 T)dx, \tag{30}$$

or

$$\Delta L = \int_0^1 \alpha_0 (T-1)dx + \int_0^1 \alpha_1 (T^2 - T)dx. \tag{31}$$

Higher order terms from equaiton (29) can be reduced by using observation time $t \gg \frac{L^2}{k} = 1$. So, the temperature can be approximated by

$$T(x,t) \approx \frac{2}{\pi}\sin \pi x\, e^{-\pi^2 t} + 1 + (t-1)x. \tag{32}$$

Substituting equation (32) into (31), yields the following integrals

$$\alpha_0 \int_0^1 (T-1)dx = \alpha_0 \int_0^1 \left(\frac{2}{\pi}\sin \pi x\, e^{-\pi^2 t} + (t-1)x\right)dx = \alpha_0\left(\frac{4}{\pi^2}e^{-\pi^2 t} + \frac{t-1}{2}\right), \tag{33}$$

And

$$\alpha_1 \int_0^1 \left(\frac{4}{\pi^2}\sin^2 \pi x\, e^{-2\pi^2 t} + \frac{4}{\pi}(1+t)\sin \pi x\, e^{-\pi^2 t} - \frac{4}{\pi}x \sin \pi x\, e^{-\pi^2 t} + (1+t)^2 - 2(1+t)x + x^2\right)dx$$
$$= \alpha_1\left(\frac{2}{\pi^2}e^{-2\pi^2 t} + \frac{4}{\pi^2}(1+t)e^{-\pi^2 t} + t^2 + \frac{4}{3}\right). \tag{34}$$

Thus, the total thermal expansion is given by

$$\Delta L(t) = \alpha_0\left(\frac{4}{\pi^2}e^{-\pi^2 t} + \frac{t-1}{2}\right) + \alpha_1\left(\frac{2}{\pi^2}e^{-2\pi^2 t} + \frac{4}{\pi^2}(1+t)e^{-\pi^2 t} + t^2 + \frac{4}{3}\right). \tag{35}$$

Equation (35) is a unique solution under the particular circumstances specified by chosen boundary conditions, such as $g(t)$ and $f(x)$. Depending on the experimental setup, several boundary functions may be chosen. This formulation (35) demonstrates how the thermal expansion coefficients α_0 and α_1 affect the total expansion, which is time-dependent. To analyze the time-dependency for the elongation, we may look at the following graphs

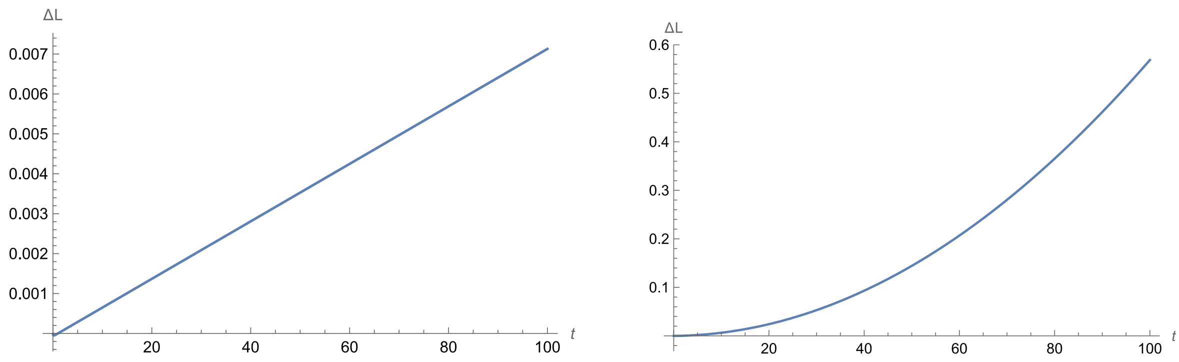

Figure 5. Time-dependent parameterization for expansion $\Delta L(t)$ with: $\alpha_0 = 0.000174$; $\alpha_1 = 0$ (left) and $\alpha_0 = 0.000174$; $\alpha_1 = 0.000056$ (right) with time $0 \leq t \leq 100$.

4.1. Parameterization: $\boldsymbol{\alpha_0 = 0.000174}$ *and* $\boldsymbol{\alpha_1 = 0}$

As shown in Figure 5 (left), the expansion curve $\Delta L(t)$ yields a linear line when $\alpha_1 = 0$. This confirms that a constant thermal expansion coefficient yields a linear elongation profile, consistent with the standard theory of linear thermal expansion. For instance, from equation (1) we have

$$\frac{dx}{dt} \sim \alpha_0 \left(\frac{dT}{dt}\right), \tag{36}$$

with a constant temperature gradient $\left(\frac{dT}{dt}\right)$, will show the same result as the figure mentioned. In this case, we have a yielding equation as

$$\Delta L(t) = \alpha_0 \left(\frac{4}{\pi^2} e^{-\pi^2 t} + \frac{t-1}{2}\right). \tag{37}$$

At $t = 0$, this simplifies to $\Delta L(t) = \frac{\alpha_0}{2}\left(\frac{8}{\pi^2} - 1\right)$. Recalling that $|\alpha_0| \ll 1$, means $\Delta L(t) \approx 0$. This indicates that the chosen boundary functions are consistent with standard laboratory fact.

4.2. Parameterization: $\boldsymbol{\alpha_0 = 0.000174}$ *and* $\boldsymbol{\alpha_1 = 0.000056}$

When the first-order coefficient satisfies $\alpha_0 > \alpha_1$, the second term of (35) can be interpreted as a correction term. The resulting non-traditional behavior in Figure 5 (right) shows deviations from the standard linear expansion model. Again, if we take the initial value $t = 0$, then we have

$$\Delta L(0) = \alpha_0 \left(\frac{4}{\pi^2} - \frac{1}{2}\right) + \alpha_1 \left(\frac{6}{\pi^2} + \frac{4}{3}\right). \tag{38}$$

With $|\alpha_1| < |\alpha_0| \ll 1$, again validating the fact this is $\Delta L(t) \approx 0$. We can see that the inclusion of the correction term increases the rate of expansion over time compared to the former one. This non-standard behavior implies that heat diffusion during thermal expansion with equipment like Figure 2, a single heat source, may be neglected, which could lead to inconsistencies in standard theory.

Incorporating temperature-dependent effects for LTEC is therefore essential for capturing the full diffusion-driven behavior of the linear thermal expansion. The LTEC from equation (35) can be determined numerically from experimental data. Overall, the method described in equation (24), combined with the exact solution from differential equation (20), provides a modified expression that is still in line with the standard theory of linear expansion for the time-dependent total expansion $\Delta L(t)$.

5. Conclusion

An alternative formulation of the thermal expansion equation in one-dimensional rods is derived using equations (7) and (8). The resulting expression for time-dependent elongation, as shown in Equation (35), is a correction to the standard linear expansion model. In order to obtain this corrected form, an inhomogeneous heat diffusion equation must be solved, which is consistent with experimental configurations frequently employed in laboratory settings. For example, a common experimental setup is to keep a steady heat source at one end of the rod while leaving the other end open to the environment or thermally insulated. Assuming that the rod initially displays a temperature distribution that may be linear, like Figure 4, observations are made for $t \geq 0$. Additional generalization is possible based on this study. To get more representative expressions for thermal expansion, other researchers could use more complicated boundary conditions and numerical evaluations of the power series integral in equation (24). These methods provide improved accuracy in simulating real-world situations with temperature-dependent diffusivity and non-uniform heating while maintaining consistency with the conventional theory of one-dimensional thermal expansion.

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