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Universal I-Love-Q Relation in Anisotropic Quark Stars within Rastall Gravity

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Abstract: This study investigates the universal I-Love-Q relation in anisotropic quark stars within the Rastall gravity framework. We employ the 4th-order Runge-Kutta numerical method to solve the slow rotation and tidal deformation equations. The analysis utilizes the MIT Bag equation of state with the Bag parameter (B) and the Color Flavor Locked (CFL) equation of state with the CFL parameter (Δ) to model the material properties of anisotropic quark stars. The universal I-Love-Q relation are explored by varying the Rastall parameter (ζ) and the anisotropic parameter (λ_H). The universal I-Love-Q relation is satisfied when variations in ζ, λ_H, B, and Δ do not affect the linear relationship between the moment of inertia (I), Love number (Λ), and quadrupole moment (Q). The universal I-Love-Q relation holds when B and Δ are varied, but breaks down when ζ and λ_H are altered.

Keyword: Anisotropic Quark Stars, Rastall Gravity, Stability, Universal Relation, Equation of State

1. Introduction

General Relativity (GR), developed by Albert Einstein in 1915, is a theory that describes how matter and energy influence space-time geometry, and vice versa, through Einstein's field equations derived from the Einstein-Hilbert Lagrangian. GR has been instrumental in understanding phenomena such as cosmological models (Peebles P. , 1993), black holes (Thorne, 1995), compact stars (Glendenning, 2012), and gravitational waves (Schilling, 2017). It has also made successful predictions, validated through experimental observations like the perihelion shift of Mercury (Turyshev, Shao, Nordtvedt, & Hellings, 2007), deflection of starlight during a solar eclipse (Dyson, Eddington, & Davidson, 1923), gravitational waves detected by LIGO (Abbott, et al., 2017), and gravitational redshift (Vessot, et al., 1980). However, GR faces challenges in certain scenarios, including the information paradox, black hole singularities (Mathur, 2009), galaxy rotation (Tian & Hsia, 2020), and the mysteries of dark energy and cosmic acceleration (Riess, et al., 1998; Perlmutter, et al., 1999).

Observational evidence of high-redshift type Ia supernovae suggests that the universe's expansion is accelerating, which has led to the introduction of dark energy to explain this phenomenon (Peebles, James, & Ratra, 2003). Dark energy, represented by the cosmological constant Λ in the Λ-CDM model, exerts negative pressure, driving accelerated expansion. While dark energy accounts for the acceleration, its exact origin remains unknown. As a result, modified gravity theories, such as Brane gravity (Randall, 2002), scalar-tensor theory (Fujii & Maeda, 2003), F(R) gravity (Kerner, 1982), and Rastall gravity (Rastall, 1972), have been proposed to address the uncertainty surrounding dark energy's source and its effects on the universe's expansion.

Rastall's theory of gravity has gained significant attention over the past few decades due to its intriguing features at both cosmological and astrophysical levels. This modified theory proposes non-conserved energy and momentum, where the divergence of the energy-momentum tensor is non-zero $\nabla_\nu T^{\nu}_{;\mu} = \zeta T_{,\mu}$, with ζ being the Rastall parameter. It generalizes General Relativity, reverting to its form when $\zeta = 0$. Recent studies, such as Cruz's work, have shown that Rastall's theory aligns with Dicke's observation, which suggests the cosmological constant decreases over time (Cruz, Lepe, & Morales-Navarrete, 2019). Additionally, Rastall's theory has been explored in various scientific areas, including black holes (Heydarzade, Moradpour, & Darabi, 2017), cosmology (Singh & Mishra, 2020), and compact stars (Abbas & Shahzad, 2020).

Recent studies on anisotropic quark stars in Rastall gravity have drawn researchers' attention to their non-uniform pressure caused by complex quark interactions. Tangphati et al. found that increasing anisotropic and Rastall parameters makes quark stars more compact and more massive than in general relativity (Tangphati, Banerjee, Hansraj, & Pradhan, 2023). Malick et al. studied the effect of charge on anisotropic quark stars within Rastall theory and found the model physically feasible and stable (Sallah & Sharif, 2025). Although anisotropic quark stars are hypothetical, there are several candidates such as PSR J1416-2230, PSR J1903+327, 4U 1820-30, Cen X-3, and EXO 1785-248 (Errehymy, Mustafa, Khedif, & Daoud, 2022). Xu Qiao suggests magnetospheric activity in pulsars can be used as an indication of the existence of quark stars, while differences in mass-radius relationships with neutron stars may help identify light-mass quark stars (Ren-xin & Guo-jun, 1998).

Nevertheless, the mass-radius relationship is highly sensitive to the star's internal structure, leading to significant discrepancies between observational data and theoretical calculations. This has motivated researchers to seek a universal relation that is insensitive to the star's internal composition. Among these researchers is Kent Yagi, who discovered the universal relation between moment of inertia I, quadrupole moment Q, and love number Λ, which is referred to as the I-Love-Q relation (Yagi & Yunes, 2013). Kent Yagi also investigated whether the anisotropic pressure affects the universal relation and he found that the anisotropy affects the universal relation only weakly (Yagi & Yunes, 2015). The I-Love-Q relations have direct applications to nuclear physics, experimental relativity and gravitational wave physics. This relationship can be used in resolving degeneracies in X-Ray pulse profiles and testing modified gravity theories (Psaltis, Özel, & Chakrabarty, 2014; Bauböck, Berti, Psaltis, & Özel, 2013).

An intriguing question is whether the universal I-Love-Q relation remains valid in the context of Rastall modified gravity on anisotropic quark stars. To address this issue, we numerically solve the equations governing slowly rotating and tidal deformation, utilizing the

MIT Bag and Color-Flavor Locked (CFL) equations of state to model quark star matter, as this model is deemed adequate for describing the fundamental interactions of quark matter (Glendenning, 2012). The anisotropic pressure is incorporated using a phenomenological approach—specifically, the Horvat model—which ensures that anisotropic effects vanish in the non-relativistic limit (Horvat, Ilijić, & Marunović, 2010). The remainder of this paper is structured as follows: Section 2 provides a comprehensive overview of Rastall gravity and introduces the theoretical framework used to derive the governing equations for slowly rotating and tidally deformed anisotropic stars. Section 3 presents the numerical calculations of the universal relations. Section 4 discusses the results and provides an analysis. Section 5 offers the conclusions. Throughout the paper, we adopt the geometric units in which $c = G = 1$.

2. Formalism

In this section, we provide an overview of Rastall gravity and present a detailed formulation of the slow-rotation and tidal deformation equations for quark stars with anisotropic matter, expanded to third order in the small spin approximation. We adopt the Hartle-Thorne method to derive the slow-rotation equations (Hartle, 1967) while the tidal deformability equation is obtained through the Hinderer method (Hinderer, 2008) .

2.1. Rastall Gravity

Rastall suggested that the conservation of energy-momentum holds true for flat spacetime, but further analysis is required to determine if it also applies to curved spacetime. He proposed that the divergence of energy-momentum tensor ($T_{\mu\nu}$) is proportional to the derivative of the Ricci scalar R. This assumption ensures the validity of the equivalence principle in General Relativity, as the Ricci scalar is zero in flat spacetime and non-zero in curved spacetime. As a result, the Rastall field equations are modified to

$$G_{\mu\nu} = 8\pi\bar{T}_{\mu\nu}, \tag{1}$$

with $G_{\mu\nu}$ is Einstens Tensor and $\bar{T}_{\mu\nu}$ is effective energy-momentum tensor, which is defined as follows

$$\bar{T}_{\mu\nu} = T_{\mu\nu} - \zeta g_{\mu\nu} T\ . \tag{2}$$

where $\boldsymbol{\zeta}$ is the Rastall parameter.

We present the energy-momentum tensor with anisotropic pressure, which is given by

$$T_{\mu\nu} = (\rho + p_{\perp})u_\mu u_\nu + p_{\perp} g_{\mu\nu} + (p - p_{\perp})k_\mu k_\nu, \tag{3}$$

Here, u_μ and k_μ represent the fluid velocity 4-vector and the radial vector, respectively. These vectors satisfy the orthonormality conditions $u_\mu u^\mu = -1$, $k_\mu k^\mu = 1$ dan $k_\mu u^\mu = 0$. Additionally, $p, p_\perp$ and ρ denote the radial pressure, tangential pressure, and energy density, respectively. By substituting equation (3) into equation (2), the resulting expression for $\bar{T}_{\mu\nu}$ equation is obtained as follows:

$$\bar{T}_{\mu\nu} = (\bar{\rho} + \bar{p}_{\perp})u_\mu u_\nu + \bar{p}_{\perp} g_{\mu\nu} + (\bar{p} - \bar{p}_{\perp})k_\mu k_\nu, \tag{4}$$

Here, $\bar{p}$, $\bar{p}_\perp$, and $\bar{\rho}$ denote the effective forms of radial pressure, tangential pressure, and energy density, respectively, and are expressed as follows:

$$\overline{\rho} = (1-\zeta)\rho + \zeta p + 2\zeta p_{\perp}, \tag{5}$$
$$\overline{p} = \zeta\rho + (1-\zeta)p - 2\zeta p_{\perp}, \tag{6}$$
$$\overline{p}_{\perp} = \zeta\rho - \zeta p_r + (1-2\zeta)p_{\perp}, \tag{7}$$

We define the anisotropic parameter as $\sigma = p - p_{\perp}$ where $\sigma = 0$ corresponds to an isotropic material. We utilize the anisotropic parameter model introduced by Horvat, which is defined as follows

$$\sigma = 2\lambda_H p \frac{M}{R}, \tag{8}$$

for a non-rotating configuration. Here, λ_H is a parameter that characterizes the degree of anisotropy within the Horvat model. The Horvat model represents a phenomenological approach to anisotropy, founded on the assumption that anisotropic effects vanish in the non-relativistic limit due to the influence of nuclear matter stress in compact stars. Therefore, this study is exclusively concerned with the Horvat model.

We assume that the radial pressure is barotropic, which implies that radial pressure is a function solely dependent on the energy density. The perturbation in the radial pressure is assumed to vanish according to the definition of new radial coordinate R. In this study, the radial pressure is modeled using the MIT Bag and Color-Flavor-Locked (CFL) models. The MIT Bag model, initially proposed by a group of theoretical physicists at the Massachusetts Institute of Technology (MIT), is founded on the assumption that quarks move freely within a confined region, referred to as a 'bag.' The presence of quarks within a volume V generates a bag energy of BV, where B is the bag constant. In addition to the bag energy, kinetic energy associated with the motion of the quarks within the bag also contributes. Both the kinetic energy and the bag energy influence the energy density and pressure in the quark matter. Based on this framework, a simple equation of state is derived, namely

$$p = \frac{1}{3}(\rho - 4B), \tag{9}$$

On the other hand, the CFL model was initially proposed by Alford, Rajagopal, and Wilczek (Alford, Rajagopal, & Wilczek, 1999). Alford and his colleagues demonstrated that, at sufficiently high densities, quarks of different colors and flavors will form Cooper pairs with identical Fermi momenta. In this phase, the quark matter transitions into the CFL phase, where it remains electrically neutral, and electrons are absent. The CFL phase is characterized by pairwise interactions, corresponding to the Cooper pairs ud, us, and ds. The equation of state for this model is provided by

$$\rho = 3p + 4B - \frac{9\alpha\mu^2}{\pi^2}, \tag{10}$$

This equation closely resembles the MIT Bag equation of state, with the addition of the condensation term $9\alpha\mu^2/\pi^2$. In the absence of the condensation term, the CFL equation of state reduces to the MIT Bag equation of state.

2.2. Slow Rotating

We begin by explaining the ansatz metric and introducing the energy-momentum tensor with anisotropic pressure. The ansatz metric employed is given by

$$ds^2 = -e^{\nu}(1 + 2\epsilon^2 h(r,\theta))dt^2 + e^{\lambda}\left(1 + 2\epsilon^2 e^{\lambda} m(r,\theta)/r\right)dr^2 + r^2\left(1 + 2\epsilon^2 k(r,\theta)\right)\left(d\theta^2 + \sin^2\theta\,\{d\phi - \epsilon[\Omega - \omega(r,\theta)]dt\}^2\right) + O(\epsilon^3). \tag{11}$$

Here, ϵ denotes an order parameter, Ω represents the angular velocity, ν and λ are the background metrics and ω, h, k, and m are the perturbation metrics. The mass function M is related to background metric λ via

$$e^{-\lambda} = 1 - 2M/r. \tag{12}$$

The perturbation metrics are expressed in terms of the Legendre polynomial function as follows:

$$\omega(r,\theta) = \omega_1(r)P_1'(\cos\theta), \tag{13}$$

$$h(r,\theta) = h_0(r) + h_2(r)P_2(\cos\theta), \tag{14}$$

$$m(r,\theta) = m_0(r) + m_2(r)P_2(\cos\theta)\,, \tag{15}$$

$$k(r,\theta) = k_2(r)P_2(\cos\theta), \tag{16}$$

where $P_l'(\cos\theta) = dP_l(\cos\theta)/d(\cos\theta)$ dan $\omega_1, h_0, h_2, m_0, m_2$ dan k_2 represent perturbation metrics, the functions of which depend on the radial coordinate r. Hartle-Thorne introduced new radial coordinates R to account for the deformation of the star from its spherical shape due to rotation. The new and old radial coordinates are related by the function $\xi(R,\theta)$, where

$$r(R,\theta) = R + \epsilon^2\xi(R,\theta) + O(\epsilon^4). \tag{17}$$

The function ξ characterizes the deformed shape of a slowly rotating star. Additionally, the functions ξ can be expressed in terms of Legendre polynomials as follows :

$$\xi(R,\theta) = \xi_0(R) + \epsilon^2\xi_2(R)P_2(\cos\theta) + O(\epsilon^4). \tag{18}$$

Furthermore, we can express anisotropic parameter using Legendre functions as shown below

$$\sigma(R,\theta) = \sigma_0(R) + \sigma_0^{(2)}(R) + \sigma_2^{(2)}(R)P_2(\cos\theta) + O(\epsilon^4) \tag{19}$$

Building upon the aforementioned approach, we proceed to derive an equation characterizing the slow rotation of an anisotropic quark star. By employing equation (1), the corresponding equations governing the $O(\epsilon^0)$ term can be obtained :

$$\frac{dM}{dR} = 4\pi R^2\bar{\rho} \tag{20}$$

$$\frac{d\nu}{dR} = 2\frac{4\pi R^3\bar{p} + M}{R(R-2M)} \tag{21}$$

$$\frac{d\bar{p}}{dR} = -\frac{(4\pi R^3\bar{p} + M)(\bar{\rho} + \bar{p})}{R(R-2M)} - \frac{2\sigma_0}{R} \tag{22}$$

While at $O(\epsilon)$, one finds

$$\frac{d^2\omega_1}{dR^2} = 4\frac{\pi R^2(\overline{\rho} + \overline{p})e^{\lambda} - 1}{R}\frac{d\omega_1}{dR} + 16\pi(\overline{\rho} + \overline{p} - \sigma_0)e^{\lambda}\omega_1\,, \tag{23}$$

and at $O(\epsilon^2)$, one finds

$$\sigma_2 = (\overline{\rho} + \overline{p}_r - \sigma_0)h_2 - \frac{\sigma_0}{R - 2\overline{M}} m_2 + \left(\frac{d\sigma_0}{dR} - \frac{d\overline{p}_r}{dR}\right)\xi_2 + \frac{R^2}{3}(\overline{\rho} + \overline{p}_r - \sigma_0)e^{-\nu}\omega_1^2\,, \tag{24}$$

$$m_2 = -Re^{-\lambda}h_2 + \frac{1}{6}R^4 e^{-(\nu+\lambda)}\left(Re^{-\lambda}\left(\frac{d\omega_1}{dR}\right)^2 + 16\pi R\omega_1^2(\overline{\rho} + \overline{p}_r - \sigma_0)\right), \tag{25}$$

$$\frac{dk_2}{dR} = -\frac{dh_2}{dR} + \frac{R - 3\overline{M} - 4\pi\overline{p}_r R^3}{R^2} e^{\lambda} h_2 + \frac{R - \overline{M} + 4\pi\overline{p}_r R^3}{R^3} e^{2\lambda} m_2\,, \tag{26}$$

$$\begin{aligned}\frac{dh_2}{dR} &= \frac{3e^{\lambda}}{R}h_2 - \frac{4\pi\overline{p}_r R^3 - \overline{M} + R}{R}e^{\lambda}\frac{dk_2}{dR} + \frac{2e^{\lambda}}{R}k_2 + \frac{8\pi\overline{p}_r R^2 + 1}{R^2}e^{2\lambda}m_2 \\ &\quad + \frac{1}{12}R^3 e^{-\nu}\left(\frac{d\omega_1}{dR}\right)^2 - 4\pi R e^{\lambda}\xi_2\frac{d\overline{p}_r}{dR}\,,\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{d\xi_2}{dR} &= \frac{1}{6R^2}\left(\frac{d\overline{p}_r}{dR}\right)^{-1}\Bigg\{6R^2(\overline{\rho} + \overline{p}_r)\frac{dh_2}{dR} + 12\sigma_0 R^2\frac{dk_2}{dR} \\ &\quad + 3\left(R^2(\overline{p}_r + \overline{\rho})\frac{d^2\nu}{dR^2} - 4\sigma_0\right)\xi_2 + 12R\sigma_2 \\ &\quad - 2R^3\left[\overline{\rho} + \overline{p}_r - \sigma_0\right]e^{-\nu}\omega_1\left(\left(R\frac{d\nu}{dR} - 2\right)\omega_1 - 2R\frac{d\omega_1}{dR}\right)\Bigg\}.\end{aligned} \tag{28}$$

This equation agrees with that for isotropic matter when $\sigma_0 = 0$.

2.3. Tidal Deformation

The spacetime geometry employed to calculate the equations governing a tidally deformed star is derived from equation (11), with the condition $\omega_1 = 0$. The analysis of tidal deformation pertains to a binary compact object system, wherein the primary object is assumed to be non-rotating and experiences deformation due to the tidal field induced by its companion.

The Love number characterizes a star's capacity to deviate from a perfect spherical shape under external tidal forces. Determining the Love number requires obtaining the asymptotic form of the exterior solution within the buffer zone—the region corresponding to the orbital separation between the stars. For instance, the (t, t) component of the metric in this buffer zone can be approximated as follows:

$$\frac{1 - g_{tt}}{2} = -\frac{M_*}{R} - \frac{Q^{(tid)}}{R^3}P_2(cos\,\theta) + \frac{1}{3}E^{(tid)}R^2 P_2(cos\,\theta) + O\left(\frac{1}{R^4}, R^3\right). \tag{29}$$

Here, $Q^{(tid)}$ and $E^{(tid)}$ correspond to the tidally induced quadrupole moment and the external tidal potential generated by the companion, respectively. Conversely, the exterior solution h_2 can be determined exactly from equations (24) - (28), where:

$$\begin{aligned}h_2^{ext} &= c_1\left(\frac{R}{M}\right)\left(1 - \frac{2M}{R}\right)\left(-\frac{2M(R-M)(3R^2 - 6MR - 2M^2)}{R^2(R-2M)^2} + 3\ln\left(\frac{R}{R-2M}\right)\right) \\ &\quad + c_2\left(\frac{R}{M}\right)^2\left(1 - \frac{2M}{R}\right),\end{aligned} \tag{30}$$

where c_1 and c_2 are constants of integration. By performing a Taylor expansion of the above expression within the buffer zone, we obtain:

$$h_2^{ext} = \frac{16}{5} c_1 \frac{M^3}{R^3} + c_2 \frac{R^2}{M^2} + O\left(\frac{1}{R^4}, R\right) \tag{31}$$

Subsequently, $Q^{(tid)}$ and $E^{(tid)}$ are determined by matching equation (31) with equation (29). The Love number is then defined as follows:

$$\Lambda = -\frac{Q^{(tid)}}{E^{(tid)}} \tag{32}$$

For a more comprehensive analysis, it is also useful to define dimensionless numbers :

$$k_2 = \frac{3}{2} \frac{\Lambda}{R^5}, \tag{33}$$

$$\overline{\Lambda} = \frac{\Lambda}{M^5}. \tag{34}$$

Upon completing the calculation, the Love number can alternatively be expressed as follows:

$$k_2 = \frac{8}{5} C^5 (1 - 2C)^2 (2C(y-1) - y + 2) \Big(2C(4(y+1)C^4 + (6y-4)C^3 + (26-22y)C^2 + 3(5y-8)C - 3y + 6) - 3(1-2C)^2 (2C(y-1) - y + 2) \log\left(\frac{1}{1-2C}\right)\Big)^{-1}, \tag{35}$$

Here, $C = M/R$ denotes the compactness of the star, and y depends on the value of h_2 and its derivatives evaluated at the stellar surface. For stars with continuous density profiles at the surface, the value of y is given by:

$$y = \frac{R h_2'}{h_2}. \tag{36}$$

A correction to y arises for stars with discontinuous density profiles at the surface, where

$$y = \frac{R_* h_2'}{h_2} + (1 - \zeta) \frac{4\pi \rho R^2 \xi_2}{h_2 (R - 2M)}. \tag{37}$$

Anisotropic quark stars exhibit surfaces with discontinuous density profiles. In this case, equation (37) is used to determine the value of y. The values of h_2 and ξ_2 in equation are obtained from the numerical solution of equations (II.61)–(II.65) by setting $\omega_1$with the numerical method detailed in the following chapter.

2.4. Universal Relations

Universal behavior refers to a tendency observed in physical systems that is independent of the internal details of those systems. The definition of universality may vary across different scientific disciplines. In statistical mechanics, for instance, universality is associated with systems belonging to large classes whose macroscopic properties are not influenced by the specific dynamics of the system. In astrophysics, the concept of universality in black holes states that the external metric tensor field of a static and isolated black hole can be fully described by only three global parameters: mass, electric charge, and angular momentum (spin). This statement is commonly referred to as the *no-hair theorem*. According to this concept, all information about matter entering a black hole becomes hidden behind the event horizon and cannot be observed by any external agent.

Unlike black holes, stars do not possess an event horizon, allowing information such as their internal composition to be observed by external agents. This internal composition influences the external gravitational field produced by the star. However, recent studies have shown that the external gravitational field of compact stars exhibits certain universal properties. One such example is the I-Love-Q relation, which connects the moment of inertia, the Love number, and the quadrupole moment. The moment of inertia measures an object's resistance to changes in rotational motion at a given angular momentum. The quadrupole moment quantifies the extent of a star's deformation due to rotation, while the Love number indicates how easily an object deforms in response to an external tidal field. This relation suggests the possible existence of a no-hair-like theorem for neutron stars or quark stars.

Within the framework of general relativity, the I-Love-Q relations are computed numerically. Kent Yagi and Nicolás Yunes discovered that these relations are independent of the equation of state. Based on this finding, they formulated a fitting equation derived from the numerical correlations, which is expressed as follows:

$$\ln y_i = a_i + b_i \ln x_i + c_i (\ln x_i)^2 + d_i (\ln x_i)^3 + e_i (\ln x_i)^4 . \quad (38)$$

with the coefficients given by Table 1

Table 1. Table of numerical coefficients for the empirical fitting of the I-Love-Q relations

y_i	x_i	a_i	b_i	c_i	d_i	e_i
$\bar{I}$	$\bar{\Lambda}$	1.496	0.05951	0.02238	-6.953×10^{-4}	8.345×10^{-6}
$\bar{I}$	$\bar{Q}$	1.393	0.5471	0.03028	0.01926	4.434×10^{-4}
$\bar{Q}$	$\bar{\Lambda}$	0.194	0.09163	0.04812	-4.283×10^{-3}	1.245×10^{-4}

where the dimensionless quantities: the normalized moment of inertia ($\bar{I}$), the dimensionless tidal deformability ($\bar{\Lambda}$), and the normalized quadrupole moment ($\bar{Q}$) are defined by

$$\bar{I} = \frac{I}{M^3}. \quad (39)$$

$$\bar{Q} = -\frac{Q\,M}{I^2 \Omega^2} \quad (40)$$

$$\bar{\Lambda} = \frac{\Lambda}{M^5} \quad (41)$$

3. Numerical Methods

In general, the differential equations are integrated from the center, r_ϵ, to the surface R_*, using the fourth-order Runge-Kutta method. To ensure numerical stability, we set r_ϵ=100 cm, which is significantly smaller than the stellar radius, typically on the order of tens of kilometers. The boundary conditions at the stellar center are derived from the analysis of the equations in the vicinity of the center. One finds :

$$\bar{\rho}(r_\epsilon) = \bar{\rho}_c + \bar{\rho}_2 r_\epsilon^2 + O(x^3). \quad (42)$$

$$M(r_\epsilon) = \frac{4\pi}{3} \bar{\rho}_c r_\epsilon^3 + O(x^5) \quad (43)$$

$$\nu(r_\epsilon) = \nu_c + \frac{4\pi}{3}(3\bar{p}_c + \bar{\rho}_c)r_\epsilon^2 + O(x^4) \tag{44}$$

The function is expanded around $x \equiv r_\epsilon/R_* \ll 1$. Here, ρ_c and p_c represent the central density and pressure, respectively, while ν_c is a constant determined by matching the interior and exterior solutions at the stellar surface. For a given central pressure p_c, the stellar radius R_* and mass M_* are obtained from the condition that the pressure vanishes at the surface, i.e., $p_r(R_*) = 0$. A range of p_c values is used to derive the mass-radius relation. The actual value of ν is obtained by matching the interior and exterior solutions at the stellar boundary. The exterior solution used in this context is given by:

$$\nu^{\text{ext}} = \ln\left(1 - \frac{2M}{R}\right). \tag{45}$$

For calculations in the slow-rotation regime, the equation is integrated with boundary conditions specified at the center of the star. These boundary conditions are derived from a series solution analysis of the differential equation, which is formulated as follows:

$$\omega_1(r_\epsilon) = \omega_{1c} + \frac{8\pi}{5}(\rho_c + p_c)\omega_{1c}r_\epsilon^2 + O(x^3)\,, \tag{46}$$

$$h_2(r_\epsilon) = C_1 r_\epsilon^2 + O(x^3)\,, \tag{47}$$

$$k_2(r_\epsilon) = -C_1 r_\epsilon^2 + O(x^3)\,, \tag{48}$$

$$\xi_2(r_\epsilon) = -\frac{e^{-\nu_c}(2\eta - 1)(p_c + \rho_c)(3e^{\nu_c}C_1 + \omega_{1c}^2)}{4\pi(3p_c^2(\eta + 2(\eta - 1)\lambda_H) + 2p_c\rho_c(3\eta - 1)(\lambda_H + 1) + (3\eta - 2)\rho_c^2)} r_\epsilon + O(x^2)\,, \tag{49}$$

The constant C_1 is determined by matching the interior and exterior solutions at the stellar surface. The boundary condition at the surface is obtained using the exterior solution of the equation. For example, the moment of inertia I and the quadrupole moment Q can be determined from the asymptotic behavior of the exterior solutions ω_1 and h_2 at spatial infinity.

$$\omega_1^{\text{ext}} = \Omega\left(1 - \frac{2I}{R^3}\right), \tag{50}$$

$$h_2^{\text{ext}} = -\frac{Q}{R^3} + O\left(\frac{1}{R^4}\right), \tag{51}$$

This solution does not depend on the anisotropic model, since the pressure vanishes at the stellar surface. The exterior solution contains an integration constant whose value is fixed by matching it with the interior solution at the surface. This constant determines the multipole moments of the gravitational field in the exterior region.

4. Results and Discussion

This section addresses two main topics. First, we analyze the configuration in the moment of inertia, quadrupole moment, and Love number for anisotropic quark stars in the framework of Rastall gravity. Second, we investigate the influence of varying the parameters ζ, λ_H, B, and Δ on the universal relations among the moment of inertia, Love number, and quadrupole moment. The universal relations for anisotropic stars in general relativity have been studied by Yagi and Yunes. They found that variations in the equation of state increase with growing anisotropy; however, the anisotropic effect is relatively weak, with differences of approximately 10% compared to the isotropic case, within the range $-2 \leq \lambda_H \leq 2$.

We compute the moment of inertia, quadrupole moment, and Love number, with the results presented in Figure 1 for the MIT Bag model. In general, the moment of inertia increases almost linearly with stellar mass in each model. Prior to reaching the maximum mass, the stellar radius begins to decrease, leading to a drop in the moment of inertia. An increase in ζ and B, tends to reduce the moment of inertia. In contrast, a larger value of λ_H increases the moment of inertia in high-density regions, while its effect is negligible in low-density regions. We conclude that increasing ζ and B, along with decreasing λ_H, results in a star that tends to rotate less easily, as indicated by a larger moment of inertia.

We present the numerical solutions of equations (20)–(28) in Figures 2, with the corresponding fitting results summarized in Table 2. The I–Love–Q relation is analyzed using dimensionless quantities: the normalized moment of inertia ($\bar{I}$), the dimensionless tidal deformability ($\bar{\Lambda}$), and the normalized quadrupole moment ($\bar{Q}$), as defined in equations (39), (40), and (41). This relation is considered universal if variations in the parameters ζ, λ_H, B, and Δ do not significantly alter the correlation among $\bar{I}$, $\bar{Q}$, and $\bar{\Lambda}$, which would otherwise form a single linear relation. Our results show that the universality of the I–Love–Q relation breaks down when ζ is varied, as the corresponding points deviate from the expected linear behavior. Similarly, variations in λ_H also lead to a loss of universality, particularly at high stellar densities, although the deviation remains negligible at low-density regimes. These findings are consistent with those reported by Yagi and Yunes (2015) in their study of anisotropic compact stars. It should be noted that the conclusions presented here are drawn for a specific equation of state, namely the MIT Bag model with $B = 60$ MeV.

It is necessary to investigate whether the internal structure of a star or variations in the equation of state affect the universality relation within a specific model framework (with fixed $\Lambda_H$and ζ), as illustrated in **Figure 3**. Interestingly, we find that the internal structure of the star does not significantly affect the universality of the I–Love–Q relation within the given model framework.This conclusion is supported by the fractional difference between the numerical results and the fitted data, which remains on the order of 10−3.

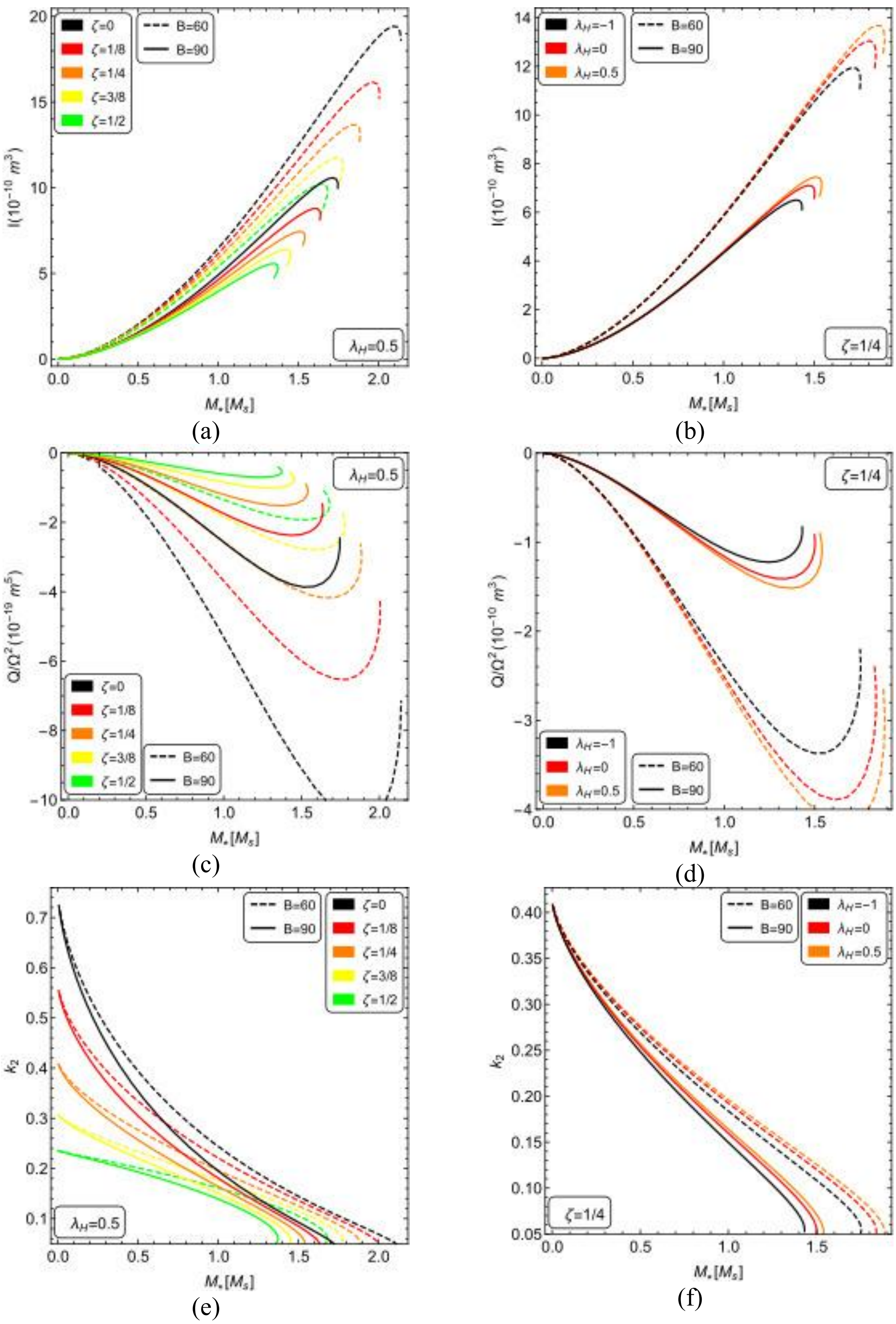
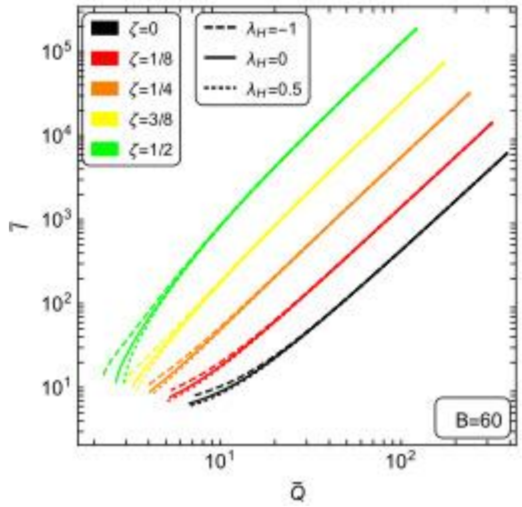
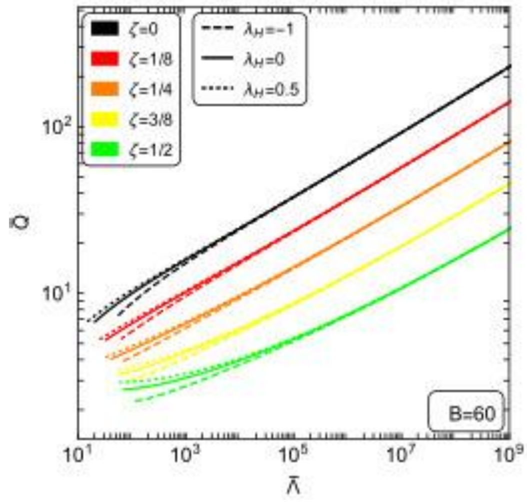


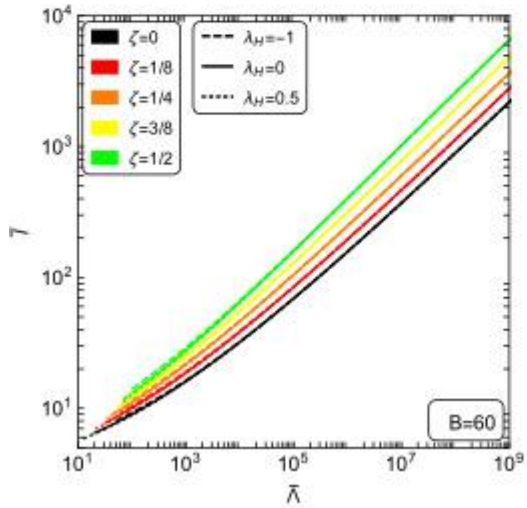
Figure 1. The relationship between stellar mass M_* and the quantities I ,Q/Ω^2 and k_2 for the MIT Bag model



(a)

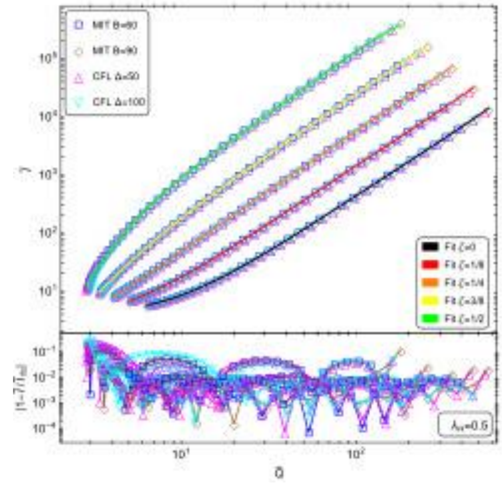


(b)

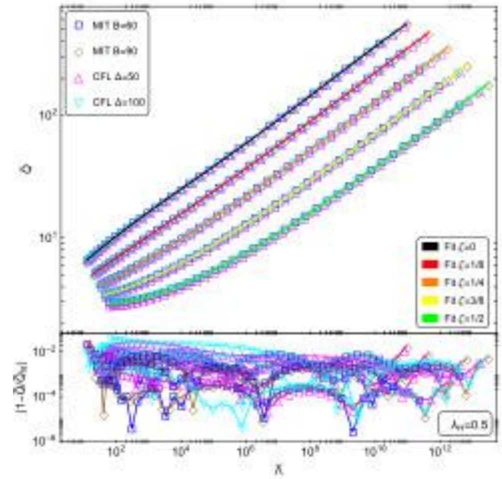


(c)

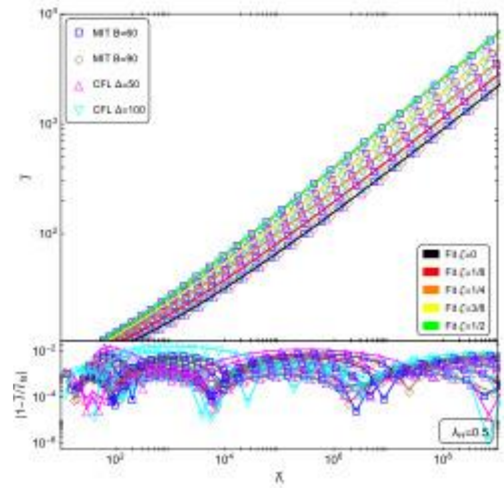
Figure 2 Relationship between the quantities $\bar{I}$, $\bar{Q}$ and $\bar{\Lambda}$ for the MIT Bag model with B=60 MeV



(a)



(b)



(c)

Figure 3 The relationship between the quantities $\bar{I}$, $\bar{Q}$ and $\bar{\Lambda}$ for λ_H=0.5 with

	various models of the equation of state

Table 2 Table of numerical coefficients for the empirical fitting of the I-Love-Q relations

λ_H	ζ	y_i	x_i	a_i	b_i	c_i	d_i	e_i
0.5	0	$\bar{I}$	$\bar{Q}$	4.1361	-3.8246	1.7807	-2.4437.E-01	1.2579.E-02
0.5	1/8	$\bar{I}$	$\bar{Q}$	1.9530	-1.5466	1.2087	-1.8195.E-01	1.0149.E-02
0.5	1/4	$\bar{I}$	$\bar{Q}$	-0.2255	1.2128	0.3914	-7.3785.E-02	4.8202.E-03
0.5	3/8	$\bar{I}$	$\bar{Q}$	-2.7778	5.2812	-1.1724	1.9285.E-01	-1.2015.E-02
0.5	1/2	$\bar{I}$	$\bar{Q}$	-5.3991	10.7225	-3.7380	7.1504.E-01	-5.0573.E-02
0.5	0	$\bar{Q}$	$\bar{\Lambda}$	1.2684	0.2690	-0.0102	5.1376.E-04	-8.6051.E-06
0.5	1/8	$\bar{Q}$	$\bar{\Lambda}$	1.0702	0.1937	-0.0024	1.5825.E-04	-2.7883.E-06
0.5	1/4	$\bar{Q}$	$\bar{\Lambda}$	1.0215	0.0992	0.0050	-1.1352.E-04	9.8085.E-07
0.5	3/8	$\bar{Q}$	$\bar{\Lambda}$	1.0834	-0.0002	0.0113	-3.0442.E-04	3.2259.E-06
0.5	1/2	$\bar{Q}$	$\bar{\Lambda}$	1.1896	-0.0848	0.0149	-3.7088.E-04	3.6498.E-06
0.5	0	$\bar{I}$	$\bar{\Lambda}$	1.4614	0.0749	0.0206	-6.1701.E-04	7.2401.E-06
0.5	1/8	$\bar{I}$	$\bar{\Lambda}$	1.4652	0.1037	0.0192	-5.8627.E-04	6.9191.E-06
0.5	1/4	$\bar{I}$	$\bar{\Lambda}$	1.4322	0.1376	0.0179	-5.6312.E-04	6.7407.E-06
0.5	3/8	$\bar{I}$	$\bar{\Lambda}$	1.3734	0.1722	0.0168	-5.4791.E-04	6.6364.E-06
0.5	1/2	$\bar{I}$	$\bar{\Lambda}$	1.3067	0.2017	0.0163	-5.5229.E-04	6.7029.E-06
0	0	$\bar{I}$	$\bar{Q}$	4.3611	-3.8976	1.7696	-2.3853.E-01	1.2079.E-02
0	1/8	$\bar{I}$	$\bar{Q}$	2.2276	-1.7074	1.2385	-1.8333.E-01	1.0085.E-02
0	1/4	$\bar{I}$	$\bar{Q}$	0.3752	0.6545	0.5873	-1.0420.E-01	6.5744.E-03
0	3/8	$\bar{I}$	$\bar{Q}$	-1.4053	3.6963	-0.5026	7.0707.E-02	-3.9045.E-03
0	1/2	$\bar{I}$	$\bar{Q}$	-2.9631	7.5141	-2.2206	4.0924.E-01	-2.8327.E-02
0	0	$\bar{Q}$	$\bar{\Lambda}$	1.1262	0.3008	-0.0128	6.0554.E-04	-9.7934.E-06
0	1/8	$\bar{Q}$	$\bar{\Lambda}$	0.9509	0.2181	-0.0043	2.2395.E-04	-3.6287.E-06
0	1/4	$\bar{Q}$	$\bar{\Lambda}$	0.8970	0.1235	0.0032	-5.3842.E-05	2.4673.E-07
0	3/8	$\bar{Q}$	$\bar{\Lambda}$	0.9102	0.0327	0.0089	-2.2927.E-04	2.3405.E-06
0	1/2	$\bar{Q}$	$\bar{\Lambda}$	0.9433	-0.0413	0.0120	-2.8704.E-04	2.7456.E-06
0	0	$\bar{I}$	$\bar{\Lambda}$	1.4908	0.0695	0.0209	-6.2785.E-04	7.3570.E-06
0	1/8	$\bar{I}$	$\bar{\Lambda}$	1.4916	0.1003	0.0194	-5.8592.E-04	6.8639.E-06
0	1/4	$\bar{I}$	$\bar{\Lambda}$	1.4666	0.1331	0.0181	-5.6483.E-04	6.7129.E-06
0	3/8	$\bar{I}$	$\bar{\Lambda}$	1.4281	0.1640	0.0172	-5.5814.E-04	6.7151.E-06
0	1/2	$\bar{I}$	$\bar{\Lambda}$	1.3964	0.1871	0.0172	-5.7653.E-04	6.9464.E-06
-1	0	$\bar{I}$	$\bar{Q}$	5.0443	-4.2136	1.7939	-2.3246.E-01	1.1331.E-02
-1	1/8	$\bar{I}$	$\bar{Q}$	2.7298	-1.9378	1.2528	-1.7753.E-01	9.3981.E-03
-1	1/4	$\bar{I}$	$\bar{Q}$	1.2234	-0.0291	0.7932	-1.3144.E-01	7.9044.E-03
-1	3/8	$\bar{I}$	$\bar{Q}$	0.2391	1.9246	0.2072	-5.3485.E-02	4.0766.E-03
-1	1/2	$\bar{I}$	$\bar{Q}$	-0.0962	3.7879	-0.4670	5.6163.E-02	-2.6190.E-03
-1	0	$\bar{Q}$	$\bar{\Lambda}$	0.6943	0.4012	-0.0214	9.2423.E-04	-1.4133.E-05
-1	1/8	$\bar{Q}$	$\bar{\Lambda}$	0.6571	0.2767	-0.0087	3.6906.E-04	-5.4062.E-06
-1	1/4	$\bar{Q}$	$\bar{\Lambda}$	0.6374	0.1707	-0.0001	4.5665.E-05	-8.8387.E-07
-1	3/8	$\bar{Q}$	$\bar{\Lambda}$	0.6011	0.0861	0.0055	-1.3040.E-04	1.2953.E-06
-1	1/2	$\bar{Q}$	$\bar{\Lambda}$	0.5186	0.0280	0.0078	-1.7712.E-04	1.6856.E-06
-1	0	$\bar{I}$	$\bar{\Lambda}$	1.5911	0.0485	0.0226	-6.8601.E-04	8.1153.E-06

-1	1/8	$\bar{I}$	$\bar{\Lambda}$	1.5635	0.0897	0.0199	-5.9550.E-04	6.8850.E-06
-1	1/4	$\bar{I}$	$\bar{\Lambda}$	1.5359	0.1249	0.0183	-5.6116.E-04	6.5448.E-06
-1	3/8	$\bar{I}$	$\bar{\Lambda}$	1.5232	0.1516	0.0177	-5.6308.E-04	6.6599.E-06
-1	1/2	$\bar{I}$	$\bar{\Lambda}$	1.5375	0.1674	0.0181	-5.9478.E-04	7.0436.E-06

5. Conclusion

This study reveals that the configuration of the moment of inertia, quadrupole moment, and Love number in anisotropic quark stars under Rastall gravity is influenced by the parameters ζ, λ_H, B, and Δ. The universality of the I–Love–Q relation breaks down when ζ and λ_H are varied, particularly at high stellar densities. However, the internal structure and the equation of state do not significantly affect this universal relation within a fixed model framework, with deviations remaining around the order of 10^{-3}. These results are consistent with findings by Yagi and Yunes, and are based on the MIT Bag model with B = 60 MeV.. The findings of this study are expected to serve as a reference for identifying anisotropic quark star candidates based on their macroscopic properties. This research can also be extended to more realistic stellar models, such as neutron stars or white dwarfs. Observational data from such objects are anticipated to provide tighter constraints on the Rastall parameters.

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