Journal of Mathematics and Mathematics Education
https://jurnal.uns.ac.id/jmme
p-ISSN 2089-8878; e-ISSN 2715-8276
Vol. 13, No. 01, June 2023

NORMAL DISTRIBUTION APPROXIMATION THROUGH BINOMIAL AND POISSON DISTRIBUTION

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Abstrak: Tujuan dari penelitian ini adalah untuk membuktikan pendekatan dari distribusi normal melalui distribusi binomial dan distribusi Poisson baik secara teoritis maupun induktif. Metode penelitian yang digunakan adalah dengan studi literatur dengan mengumpulkan berbagai pustaka yang berkaitan dengan masalah yang diteliti. Secara teoritis distribusi binomial dan distribusi Poisson akan semakin mendekati distribusi normal ketika n mendekati tak hingga atau dengan kata lain limit n menuju tak hingga dari distribusi binomial atau distribusi Poisson akan mendekati distribusi normal. Secara teoritis pendekatan tersebut dapat dibuktikan dengan metode pembangkit momen. Sedangkan secara induktif, pendekatan tersebut dapat dilihat dari poligon distribusi binomial dan distribusi Poisson yang akan semakin mendekati kurva normal ketika jumlah datanya sangat besar mendekati tak hingga.
Kata kunci : D*istribusi normal, distribusi binomial, distribusi Poisson, metode pembangkit momen*

Abstract: The purpose of this research is to prove the approach of normal distribution through binomial distribution and Poisson distribution both theoretically and inductively. The research method used is a literature study by collecting various literature related to the problem under study. Theoretically, the binomial distribution and the Poisson distribution will be closer to the normal distribution when n approaches infinity or in other words the limit n towards infinity of the binomial distribution or Poisson distribution will approach the normal distribution. Theoretically, this approach can be proven by the method of generating moments. While inductively, the approach can be seen from the polygons of the binomial distribution and Poisson distribution which will increasingly approach the normal curve when the amount of data is very large approaching infinity.
Keywords: *Normal distribution, binomial distribution, Poisson distribution, moment generating method*

INTRODUCTION

Mathematics is a very important field of science in everyday life, where almost all aspects of human life are inseparable from mathematics. One branch of mathematics that is very important to master is statistics, especially for students or researchers as a support for their research. Statistics is

Accepted: June 01, 2023 Approved: June 14, 2023 Published: June 14, 2023
DOI: 10.20961/jmme.v13i1.75051

knowledge related to how to organize data, present data, and draw conclusions about a whole (called population) based on data that exists in part of the whole. Part of the whole (population) is called a sample (Budiyono,2009). One of the most basic sub-materials in statistics is special probability distributions. Special probability distributions are divided into two types, namely discrete probability distributions and continuous probability distributions. Furthermore, discrete probability distributions are further divided into several types, including the binomial distribution, Bernoulli distribution, Poisson distribution, and many more. While the continuous probability distribution is divided into normal distribution, exponential distribution, and many more. In its application, the most commonly used probability distributions are the binomial distribution, Poisson distribution and normal distribution, therefore in this discussion we will only discuss these three distributions. The binomial distribution is one of the discrete probability distributions of the number of successes in mutually independent n experiments (trials) that are mutually independent, where for each outcome of the experiment has a probability of θ. In statistics, the binomial distribution is the basis of the binomial test for statistical significance. This distribution is often used to model the number of successes in a sample size n from a population of N. Binomial distribution is when samples are independent and sampling is done with returns. A random variable X with binomial distribution is denoted $b(x;\ n,\theta)$ defined by:

$$b(x;n,\theta) = f(x) = \begin{cases} \binom{n}{x}\theta^x(1-\theta)^{n-x}, x = 1,2,3,\ldots n \\ 0 \qquad\qquad\qquad\quad ,\quad x\ other \end{cases}$$

n and θ is a parameter,

$x =$ number of successful events,

$n =$ the number of trials,

$\theta =$ the probability of success in a single experiment.

Theorem 1.1

Random variable X is binomially distributed with parameters n and θ then:

1. $M_x(t) = (\theta e^t + (1-\theta))^n$, $M_x(t) =$ moment generating function
2. $\mu = n\theta$, $\mu =$ mean
3. $\sigma^2 = n\theta(1-\theta)$, $\sigma^2 =$ variance

If the population N is larger, researchers often use the Poisson distribution or the normal distribution and rarely use the binomial distribution. The Poisson distribution it self is often referred to as the binomial distribution with very large populations. N However, there is rarely a book that shows the proof of this, so most readers just accept the information raw without knowing the guarantee of truth. A discrete random variable with Poisson distribution is defined by:

$$p(x;\lambda) = f(x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!}, x = 0,1,2,3, \dots \\ 0 \quad , x \; other \end{cases}$$

λ is a parameter,

$\lambda =$ the expected number of successful events $= n\theta$,

$\theta =$ chance of a successful event,

$n =$ the number of trials,

$x =$ number of successful events,

$e \approx 2.718 =$ a mathematical constant called the original logarithm principal,

Theorem 1.2

Random variable X is Poisson distributed with parameters λ, then:

1. $M_x(t) = e^{\lambda(e^t-1)}$, $M_x(t) =$ moment generating function
2. $\mu = \lambda$, $\mu =$ mean
3. $\sigma^2 = \lambda$, $\sigma^2 =$ variance

The normal distribution is a continuous probability distribution, unlike the binomial distribution and Poisson distribution which are discrete probability distributions. It has a bell-like graph. The normal distribution is often used invarious fields of statistics, for example, the sampling distribution of the mean will be close to normal, even though the population distribution is not a normal distribution. In addition, the normal distribution is also often used invarious distributions in statistics, where hypothesis testing often needs to assume normality of the data. A random variable X is said to have a normal distribution, denoted by $X \sim N(\mu, \sigma^2)$ if its density function is of the form:

$$X \sim N(\mu, \sigma^2) = f(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, -\infty < x < \infty \\ 0 \qquad , x \text{ other} \end{cases}$$

μ and σ^2 is parameters,

$\mu =$ mean mean of population,

$\sigma^2 =$ variance of population,

$\sigma^2 =$ standard deviation of population,

$e \approx 2.718$ is a constant,

$\pi \approx 3.141$ is a constant,

$x =$ the value of a continuous ramdom variable that satisfies $-\infty < x < \infty$

If the random variable X is normally distributed has $\mu = 0$ and $\sigma^2 = 1$, then it is defined by the standard normal distribution which is denoted by $X \sim N(0,1)$ or also called distribution Z. The density function of the standard normal distribution is as follows

$$X \sim N(0,1) = f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x)^2}, -\infty < x < \infty \\ 0 \qquad , x \text{ other} \end{cases}$$

Theorem 1.3

Random variable X is normally distributed $N(\mu, \sigma^2)$ with parameters μ and σ^2, then:

1. $M_x(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$, $M_x(t) =$ moment generating function
2. $\mu = \mu$, $\mu =$ mean
3. $\sigma^2 = \sigma^2$, $\sigma^2 =$ variance

In practice, computing and analyzing data from statistical data is not easy, especially when the data obtained is very large and will take a long time if computed manually. Therefore, Minitab software was developed to help process statistical data. Minitab software has various features that can be used for data entry, data manipulation, graph making, and various other statistical analyses.

From the description above, the question arises "what if the population size in the binomial distribution is very large?" and "can one particular distribution be formed by utilizing another distribution?". These questions then underlie the researcher to examine the relationship of the three distributions, namely the binomial distribution, Poisson distribution, and normal distribution. From the description above, the researcher took the title "Normal Distribution Approach through Binomial Distribution and Poisson Distribution".

RESEARCH METHOD

This research is a literature study, with the steps of the procedure used are first collecting some literature related to the normal distribution approach through the binomial distribution and Poisson distribution, second studying the definitions and theorems from the literature that has been collected, third proving the normal distribution approach through the binomial distribution and Poissson distribution both theoretically and inductively, and finally drawing conclusions.

RESULTS AND DISCUSSION

1. Proving Normal Distribution Approximation through Binomial Distribution and Poisson Distribution Theoretically

a. Proof of Poisson Distribution Approximation through Binomial Distribution

As explained in the theoretical study section, for the binomial case, which has very small probabilities close to 0 and n very large in other words n approaching infinity, then we can use the Poisson distribution approach to calculate the computation.

Consider the following proof of Poisson distribution approximation through binomial distribution

Unknown:

Binomial distribution $b(x; n; \theta)$

$x = 0, 1, 2, 3 \ldots$

n Very big $(n \rightarrow \infty)$

Chances of success $= \theta$

Because $n \rightarrow \infty$ an infinite limit of the binomial distribution function is required, it can be written as follows

$$\lim_{n\to\infty} b(x;n;\theta) = \lim_{n\to\infty} \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

Because θ is the probability of success, we can view θ as the average number of successful events (let λ) divided by the number of events (n).

Obtained from $\theta = \frac{\lambda}{n}$

Consider the following limit

$$\lim_{n\to\infty} \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} = \lim_{n\to\infty} \left(\frac{n!}{x!(n-x)!}\right) \frac{\lambda^x}{n^x} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \left(\frac{n(n-1)(n-2)...(n-x+1)(n-x)!}{x!(n-x)!}\right) \frac{\lambda^x}{n^x} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \left(\frac{n(n-1)(n-2)...(n-k+2)(n-x+1)}{x!}\right) \frac{\lambda^x}{n^x} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)...(n-k+2)(n-x+1)}{n^x} \frac{\lambda^x}{x!} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n}{n} \frac{(n-1)}{n} \frac{(n-2)}{n} ... \frac{(n-x+2)}{n} \frac{(n-x+1)}{n} \frac{\lambda^x}{x!} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

Let

$$u = \frac{n}{n} \frac{(n-1)}{n} \frac{(n-2)}{n} ... \frac{(n-x+2)}{n} \frac{(n-x+1)}{n}$$

$$v = \frac{\lambda^x}{x!} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

Then we can view it as $\lim_{n\to\infty} uv = \lim_{n\to\infty} u \lim_{n\to\infty} v$

Watch $\lim_{n\to\infty} u$

$$\lim_{n\to\infty} u = \lim_{n\to\infty} 1 \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) ... \left(1-\frac{x+2}{n}\right)\left(1-\frac{x+1}{n}\right)$$

Recall thaf $\lim_{n\to\infty} \frac{1}{n} = 0$, then we get

$$\lim_{n\to\infty} 1 \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) ... \left(1-\frac{x+2}{n}\right)\left(1-\frac{x+1}{n}\right)$$

$$= 1(1-0)(1-0)...(1-0)(1-0) = 1$$

Watch $\lim_{n\to\infty} v$

$$\lim_{n\to\infty} v = \lim_{n\to\infty} \frac{\lambda^x}{x!} \left(1-\frac{\lambda}{n}\right)^{n-x} = \frac{\lambda^x}{x!} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n (1-0)^{-x} = \frac{\lambda^x}{x!} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n$$

Based on the infinite limit theorem of special functions

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$ and $\lim_{n\to\infty} \left(1+\frac{m}{n}\right)^n = e^m$, then obtained

$$\frac{\lambda^x}{x!}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = \frac{\lambda^x}{x!}\lim_{n\to\infty}\left(1+\frac{(-\lambda)}{n}\right)^n = \frac{\lambda^x}{x!}e^{-\lambda} = \frac{\lambda^x e^{-\lambda}}{x!}$$

So we get $\lim_{n\to\infty} u \lim_{n\to\infty} v = 1\left(\frac{\lambda^x e^{-\lambda}}{x!}\right) = \frac{\lambda^x e^{-\lambda}}{x!}$.

We can see that $\frac{\lambda^x e^{-\lambda}}{x!}$ Is a function of the Poisson distribution

$$p(x;\lambda) = f(x) = \frac{\lambda^x e^{-\lambda}}{x!}; x = 0, 1, 2, 3 \ldots$$

b. Proof of Normal Distribution Approximation through Binomial Distribution

Suppose a random variable X with binomial distribution $b(x;n,\theta) = f(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}, x = 1,2,3,\ldots n$. Based on theorem 1.1, the binomial distribution function has a moment generating function $M_x(t) = E[e^{tx}] = (\theta e^t + (1-\theta))^n$.

To approach the normal distribution, the random variable X must be transformed into a standard normal distributed random variable Z with $Z = \frac{X-\mu}{\sigma}$.

It is known that the binomial distribution has a mean $\mu = n\theta$ and variance $\sigma^2 = n\theta(1-\theta) \longrightarrow \sigma = \sqrt{n\theta(1-\theta)}$, then $Z = \frac{X-n\theta}{\sqrt{n\theta(1-\theta)}}$ is a standard normal distributed random variable. Proof of the normal distribution approach through the binomial distribution can be done by the moment generating function method, namely by giving a limit to infinity to the moment generating function of the binomial distribution with random variables Z.

Consider the limit to infinity of the following moment generating function

$$\lim_{n\longrightarrow\infty} M_z(t) = \lim_{n\longrightarrow\infty} E[e^{tz}] = \lim_{n\longrightarrow\infty} E\left[e^{t\left(\frac{X-n\theta}{\sqrt{n\theta(1-\theta)}}\right)}\right]$$

$$= \lim_{n\longrightarrow\infty} E\left[e^{t\left(\frac{X}{\sqrt{n\theta(1-\theta)}}\right)-t\left(\frac{n\theta}{\sqrt{n\theta(1-\theta)}}\right)}\right]$$

$$= \lim_{n\longrightarrow\infty} E\left[e^{t\left(\frac{X}{\sqrt{n\theta(1-\theta)}}\right)}e^{-t\left(\frac{n\theta}{\sqrt{n\theta(1-\theta)}}\right)}\right]$$

Notice, $e^{-t\left(\frac{n\theta}{\sqrt{n\theta(1-\theta)}}\right)}$ is a constant-valued function, based on the mathematical expected value theorem that $E[cX] = cE[X]$, then obtained

$$\lim_{n\longrightarrow\infty} E\left[e^{t\left(\frac{X}{\sqrt{n\theta(1-\theta)}}\right)}e^{-t\left(\frac{n\theta}{\sqrt{n\theta(1-\theta)}}\right)}\right] = \lim_{n\longrightarrow\infty} e^{-t\left(\frac{n\theta}{\sqrt{n\theta(1-\theta)}}\right)}E\left[e^{t\left(\frac{X}{\sqrt{n\theta(1-\theta)}}\right)}\right]$$

Note that for $E\left[e^{t\left(\frac{X}{\sqrt{n\theta(1-\theta)}}\right)}\right]$ is the moment generating function of binomial distribution with random variable X, then obtained

$$E\left[e^{t\left(\frac{X}{\sqrt{n\theta(1-\theta)}}\right)}\right]=E\left[e^{\left(\frac{t}{\sqrt{n\theta(1-\theta)}}\right)X}\right]=\left(\theta e^{\left(\frac{t}{\sqrt{n\theta(1-\theta)}}\right)}+(1-\theta)\right)^n$$

Furthermore, from $\lim_{n\to\infty} e^{-t\left(\frac{n\theta}{\sqrt{n\theta(1-\theta)}}\right)}E\left[e^{t\left(\frac{X}{\sqrt{n\theta(1-\theta)}}\right)}\right]$ obtained

$$\lim_{n\to\infty} e^{-t\left(\frac{n\theta}{\sqrt{n\theta(1-\theta)}}\right)}\left(\theta e^{\left(\frac{t}{\sqrt{n\theta(1-\theta)}}\right)}+(1-\theta)\right)^n$$

$$=\lim_{n\to\infty}\left(e^{-t\left(\frac{\theta}{\sqrt{n\theta(1-\theta)}}\right)}\right)^n\left(\theta e^{\left(\frac{t}{\sqrt{n\theta(1-\theta)}}\right)}+(1-\theta)\right)^n$$

$$=\lim_{n\to\infty}\left[\left(e^{-t\left(\frac{\theta}{\sqrt{n\theta(1-\theta)}}\right)}\right)\left(\theta e^{\left(\frac{t}{\sqrt{n\theta(1-\theta)}}\right)}+(1-\theta)\right)\right]^n$$

$$=\lim_{n\to\infty}\left[\theta e^{\left(\frac{t}{\sqrt{n\theta(1-\theta)}}\right)-t\left(\frac{\theta}{\sqrt{n\theta(1-\theta)}}\right)}+(1-\theta)e^{-t\left(\frac{\theta}{\sqrt{n\theta(1-\theta)}}\right)}\right]^n$$

$$=\lim_{n\to\infty}\left[\theta e^{\left(\frac{t}{\sqrt{n\theta(1-\theta)}}\right)(1-\theta)}+(1-\theta)e^{-t\left(\frac{\theta}{\sqrt{n\theta(1-\theta)}}\right)}\right]^n$$

Note that for $e^{\left(\frac{t}{\sqrt{n\theta(1-\theta)}}\right)(1-\theta)}$, using the Taylor series theorem, $e^{\left(\frac{t}{\sqrt{n\theta(1-\theta)}}\right)(1-\theta)}$ can be transformed into

$$e^{\left(\frac{t}{\sqrt{n\theta(1-\theta)}}\right)(1-\theta)}=1+t\left(\frac{t}{\sqrt{n\theta(1-\theta)}}\right)(1-\theta)+\frac{t^2(1-\theta)^2}{2!\left(n\theta(1-\theta)\right)}+\frac{t^3(1-\theta)^3}{3!\left(\sqrt{n\theta(1-\theta)}\right)^3}+\cdots$$

Also note that for $e^{-t\left(\frac{\theta}{\sqrt{n\theta(1-\theta)}}\right)}$, using the Taylor series theorem, $e^{-t\left(\frac{\theta}{\sqrt{n\theta(1-\theta)}}\right)}$ can be transformed into

$$e^{-t\left(\frac{\theta}{\sqrt{n\theta(1-\theta)}}\right)}=1-t\left(\frac{\theta}{\sqrt{n\theta(1-\theta)}}\right)+\frac{t^2\theta^2}{2!\left(n\theta(1-\theta)\right)}-\frac{t^3\theta^3}{3!\left(\sqrt{n\theta(1-\theta)}\right)^3}+\cdots$$

So that for $\theta e^{\left(\frac{t}{\sqrt{n\theta(1-\theta)}}\right)(1-\theta)}+(1-\theta)e^{-t\left(\frac{\theta}{\sqrt{n\theta(1-\theta)}}\right)}$ obtained

$$\theta\left(1+t\left(\frac{t}{\sqrt{n\theta(1-\theta)}}\right)(1-\theta)+\frac{t^2(1-\theta)^2}{2!\left(n\theta(1-\theta)\right)}+\frac{t^3(1-\theta)^3}{3!\left(\sqrt{n\theta(1-\theta)}\right)^3}+\cdots\right)+$$

$$(1-\theta)\left(1-t\left(\frac{\theta}{\sqrt{n\theta(1-\theta)}}\right)+\frac{t^2\theta^2}{2!\left(n\theta(1-\theta)\right)}-\frac{t^3\theta^3}{3!\left(\sqrt{n\theta(1-\theta)}\right)^3}+\cdots\right)$$

$$=1+\left(\frac{t^2\theta(1-\theta)\left(\theta+(1-\theta)\right)}{2!\left(n\theta(1-\theta)\right)}\right)+\left(\frac{t^3\theta(1-\theta)\left((1-\theta)^2-\theta^2\right)}{3!\left(\sqrt{n\theta(1-\theta)}\right)^3}\right)+\cdots$$

$$=1+\left(\frac{t^2\left(\theta+(1-\theta)\right)}{2!(n)}\right)+\left(\frac{t^3\theta(1-\theta)\left((1-\theta)^2-\theta^2\right)}{3!\left(\sqrt{n\theta(1-\theta)}\right)^3}\right)+\cdots$$

Furthermore, it is obtained

$$\lim_{n\to\infty}\left[\theta e^{\left(\frac{t}{\sqrt{n\theta(1-\theta)}}\right)(1-\theta)}+(1-\theta)e^{-t\left(\frac{\theta}{\sqrt{n\theta(1-\theta)}}\right)}\right]^n$$

$$=\lim_{n\to\infty}\left[1+\left(\frac{t^2\left(\theta+(1-\theta)\right)}{2!\,(n)}\right)+\left(\frac{t^3\theta(1-\theta)\left((1-\theta)^2-\theta^2\right)}{3!\left(\sqrt{n\theta(1-\theta)}\right)^3}\right)+\cdots\right]^n$$

Let $\psi(n)=\left(\frac{t^3\theta(1-\theta)\left((1-\theta)^2-\theta^2\right)}{3!\left(\sqrt{n\theta(1-\theta)}\right)^3}\right)+\cdots$ and $\{\psi(n), n\geq 1\}$ are rows of real numbers.

Thus obtained

$$\lim_{n\to\infty}\left[1+\left(\frac{t^2}{2!\,(n)}\right)+\psi(n)\right]^n$$

From the limit of a real number sequence above, there is a lemma "suppose $\{\psi(n), n\geq 1\}$ is a sequence of real numbers such that $\lim_{n\to\infty}\psi(n)=0$. Then $\lim_{n\to\infty}\left[1+\left(\frac{a}{n}\right)+\frac{\psi(n)}{n}\right]^{bn}=e^{ab}$, with a and b are independent of n.”.

Based on the above lemma, it is obtained

$$\lim_{n\to\infty}\left[1+\left(\frac{t^2}{2!\,(n)}\right)+\psi(n)\right]^n=e^{\frac{t^2}{2}}$$

Consider $e^{\frac{t^2}{2}}$ is the moment generating function of the standard normal distribution $N(0,1)$.

So, it is evident that the binomial distribution will approach the normal distribution when the number of data is infinitely large.

c. Proof of Normal Distribution Approximation through Poisson Distribution

Suppose a random variable X which is Poisson distributed $p(x;\lambda) = f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, x = 0,1,2,3, ...$ Based on theorem 1.2, the Poisson distributed function has a moment generating function $M_x(t) = E[e^{tx}] = e^{\lambda(e^t-1)}$.

To approach the normal distribution, the random variable X must be transformed into a standard normal distributed random variable Z with $Z = \frac{X-\mu}{\sigma}$.

It is known that the Poisson distribution has a mean $\mu = \lambda$ and variance $\sigma^2 = \lambda \longrightarrow \sigma = \sqrt{\lambda}$, then $Z = \frac{X-\lambda}{\sqrt{\lambda}}$ is a standard normal distributed random variable. Proof of the normal distribution approach through the Poisson distribution can be done by the moment generating function method, namely by giving a limit to infinity to the moment generating function of the Poisson distribution with random variables Z.

Consider the limit to infinity of the following moment generating function:

$$\lim_{\lambda\to\infty} M_z(t) = \lim_{\lambda\to\infty} E[e^{tz}] = \lim_{\lambda\to\infty} E\left[e^{t\left(\frac{X-\lambda}{\sqrt{\lambda}}\right)}\right] = \lim_{\lambda\to\infty} E\left[e^{t\left(\frac{X}{\sqrt{\lambda}}\right)-t\left(\frac{\lambda}{\sqrt{\lambda}}\right)}\right]$$

$$= \lim_{\lambda\to\infty} E\left[e^{t\left(\frac{X}{\sqrt{\lambda}}\right)-t\sqrt{\lambda}}\right] = \lim_{\lambda\to\infty} E\left[e^{t\left(\frac{X}{\sqrt{\lambda}}\right)}e^{-t\sqrt{\lambda}}\right]$$

Notice, $e^{-t\sqrt{\lambda}}$ is a constant-valued function, based on the mathematical expected value theorem that $E[cX] = cE[X]$, then obtained

$$\lim_{\lambda\to\infty} E\left[e^{t\left(\frac{X}{\sqrt{\lambda}}\right)}e^{-t\sqrt{\lambda}}\right] = \lim_{\lambda\to\infty} e^{-t\sqrt{\lambda}}E\left[e^{t\left(\frac{X}{\sqrt{\lambda}}\right)}\right]$$

Note that for $E\left[e^{t\left(\frac{X}{\sqrt{\lambda}}\right)}\right]$ s the moment generating function of Poisson distribution with random variable$X$, so that it is obtained

$$E\left[e^{t\left(\frac{X}{\sqrt{\lambda}}\right)}\right] = E\left[e^{\frac{t}{\sqrt{\lambda}}X}\right] = e^{\lambda\left(e^{\frac{t}{\sqrt{\lambda}}}-1\right)}$$

Furthermore, from $\lim_{\lambda\to\infty} e^{-t\sqrt{\lambda}}E\left[e^{t\left(\frac{X}{\sqrt{\lambda}}\right)}\right]$ obtained

$$\lim_{\lambda\to\infty} e^{-t\sqrt{\lambda}}e^{\lambda\left(e^{\frac{t}{\sqrt{\lambda}}}-1\right)} = \lim_{\lambda\to\infty} e^{\lambda\left(e^{\frac{t}{\sqrt{\lambda}}}-1\right)-t\sqrt{\lambda}}$$

Note that for $e^{\frac{t}{\sqrt{\lambda}}}$, using the Taylor series theorem, $e^{\frac{t}{\sqrt{\lambda}}}$ can be transformed into

$$e^{\frac{t}{\sqrt{\lambda}}} = 1 + \frac{t}{\sqrt{\lambda}} + \frac{\left(\frac{t}{\sqrt{\lambda}}\right)^2}{2!} + \frac{\left(\frac{t}{\sqrt{\lambda}}\right)^3}{3!} + \cdots = 1 + \frac{t}{\sqrt{\lambda}} + \frac{t^2}{2!\,\lambda} + \frac{t^3}{3!\,\lambda^{\frac{3}{2}}} + \cdots$$

So that for $\lambda(e^{\frac{t}{\sqrt{\lambda}}} - 1)$ obtained

$$\lambda\left[\left(1 + \frac{t}{\sqrt{\lambda}} + \frac{t^2}{2!\,\lambda} + \frac{t^3}{3!\,t^{\frac{3}{2}}} + \cdots\right) - 1\right] = \lambda\left(\frac{t}{\sqrt{\lambda}} + \frac{t^2}{2!\,\lambda} + \frac{t^3}{3!\,\lambda^{\frac{3}{2}}} + \cdots\right)$$

$$= t\sqrt{\lambda} + \frac{t^2}{2!} + \frac{t^3}{3!\,\lambda^{\frac{1}{2}}} + \cdots$$

Furthermore, from $\lim\limits_{\lambda\to\infty} e^{\lambda\left(e^{\frac{t}{\sqrt{\lambda}}}-1\right)-t\sqrt{\lambda}}$ obtained

$$\lim_{\lambda\to\infty} e^{\lambda\left(e^{\frac{t}{\sqrt{\lambda}}}-1\right)-t\sqrt{\lambda}} = \lim_{\lambda\to\infty} e^{\left(t\sqrt{\lambda}+\frac{t^2}{2!}+\frac{t^3}{3!\lambda^{\frac{1}{2}}}+\cdots\right)-t\sqrt{\lambda}} = \lim_{\lambda\to\infty} e^{\left(\frac{t^2}{2!}+\frac{t^3}{3!\lambda^{\frac{1}{2}}}+\cdots\right)}$$

$$= \lim_{\lambda\to\infty} e^{\frac{t^2}{2!}} e^{\left(\frac{t^3}{3!\lambda^{\frac{1}{2}}}+\frac{t^4}{4!\lambda^{\frac{2}{2}}}+\cdots\right)} = e^{\frac{t^2}{2!}} \lim_{\lambda\to\infty} e^{\left(\frac{t^3}{3!\lambda^{\frac{1}{2}}}+\frac{t^4}{4!\lambda^{\frac{2}{2}}}+\cdots\right)} = e^{\frac{t^2}{2!}}(e^0) = e^{\frac{t^2}{2}}$$

Consider $e^{\frac{t^2}{2}}$ is the moment generating function of the standard normal distribution $N(0,1)$.

So, it is evident that the Poisson distribution will approach the normal distribution when the number of data is infinitely large.

2. Inductive Proof of Normal Distribution Approximation through Binomial Distribution and Poisson Distribution Using Minitab

Inductive proof is proof by using several different cases, then from these cases generalized to get the same conclusion. Proving the normal distribution approach through binomial distribution and Poisson distribution inductively can be seen through the curve. To visualize the curve of a binomial or Poisson distributed function, it will be very difficult if done manually, especially if the data is very large, therefore a software is needed to help visualize the curve. In this problem, the author chose Minitab software to help prove the normal distribution approach through binomial distribution and Poisson distribution inductively.

In this proof, according to the previous theory which states that the binomial distribution and Poisson distribution will be closer to the normal distribution when N is very large, it will be compared for N which is small and then enlarged until the curve approaches the normal curve. Because to prove this approach requires a lot of data, it will be difficult if you have to collect real data, therefore a lot of data can be made by generating data using Minitab software then comparing the curve with the normal distribution.

Furthermore, to prove the normal distribution approach through binomial distribution and Poisson distribution as follows:

a. Simulation study of normal distribution approximation through binomial distribution using Minitab software.

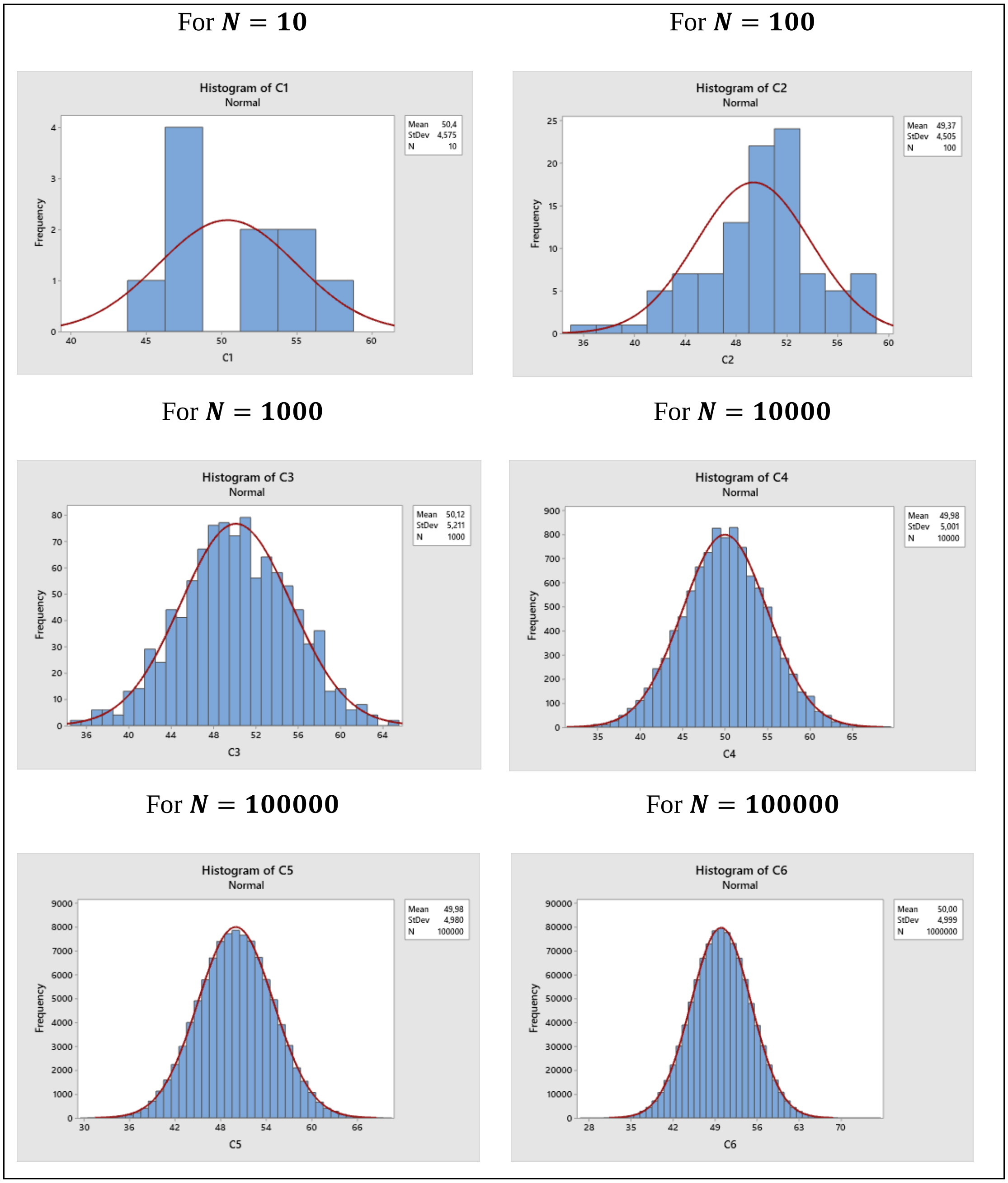


Figure 1. Normal distribution approximation through binomial distribution

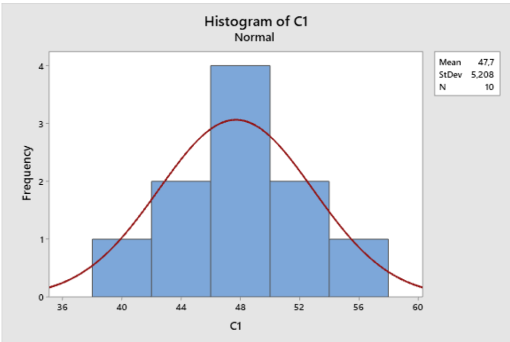
We will try to generate binomially distributed data with a small N small and then enlarged and compare the polygons with the normal curve. In this case θ will be made the same, namely 0.5. From the polygons on **Figure 1**, it can be seen that the greater the value of the N with θ fixed, the polygons are closer to a normal curve. It can also be seen that the mean is close to 50, which indicates that the binomial distribution is close to the Poisson distribution with $\lambda = n\theta = (100)(0.5) = 50$. So, it is evident that the binomial distribution polygon will approach a normal curve when the data is very large or approaches infinity.

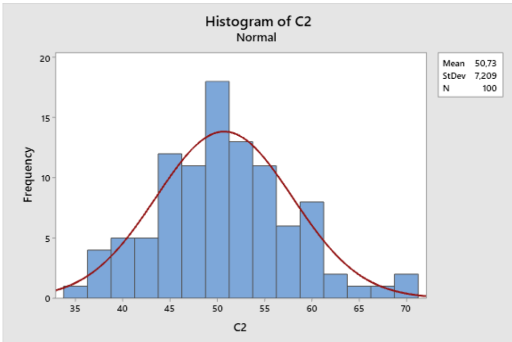
b. Simulation study of normal distribution approach through Poisson distribution using Minitab software.

We will try to generate Poisson distributed data by generalizing the binomial distribution data above, namely with N small and then enlarged and with an average $\lambda = n\theta = (100)(0.5) = 50$ then compare the polygon with the normal curve.

For $\boldsymbol{N = 10}$

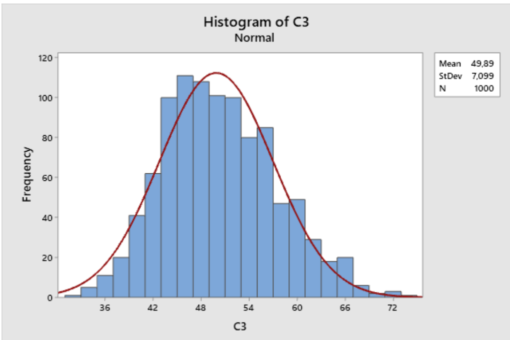
For $\boldsymbol{N = 100}$

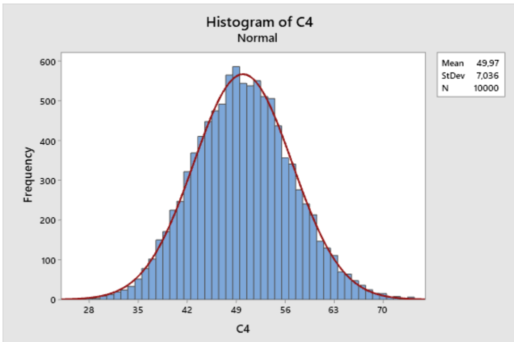




For $\boldsymbol{N = 1000}$

or $\boldsymbol{N = 10000}$





For $\boldsymbol{N = 100000}$

For $\boldsymbol{N = 100000}$

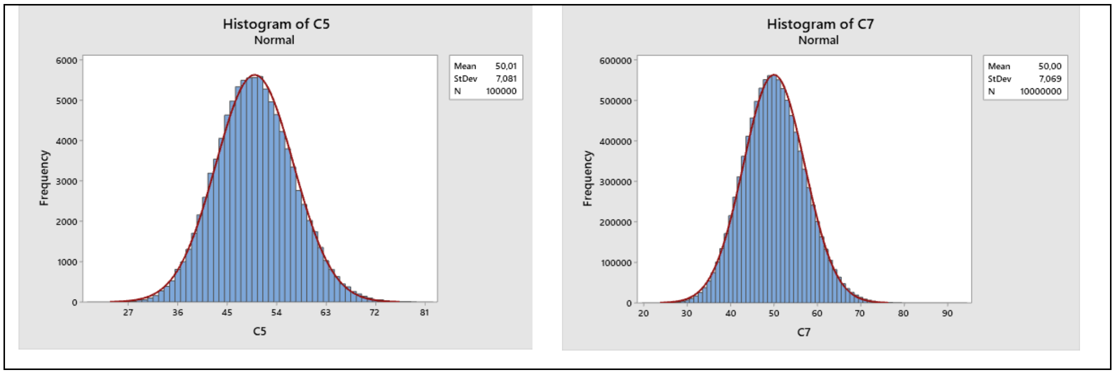


Figure 2. Normal distribution approach through Poisson distribution

From the polygons on Figure 2, it can be seen that the greater the value of the N with λ fixed, the closer the polygons are to a normal curve. So, it is evident that the Poisson distribution polygon will approach a normal curve when the data is infinitely large.

CONCLUSIONS AND SUGGESTIONS

Based on the results of the discussion, the normal distribution approach through binomial distribution and through Poisson distribution can be proven theoretically and inductively. Theoretically, the binomial distribution and Poisson distribution will be close to the normal distribution by applying limits n approaching infinity of the moment generating function of the binomial distribution and Poisson distribution which will then obtain the moment generating function of the standard normal distribution. While inductively, the normal distribution approach through binomial distribution and Poisson distribution can be done by comparing the normal curve with polygons of binomial distribution or Poisson distribution that has a large amount of data. By utilizing Minitab software, we can visualize the normal curve and polygons of the binomial distribution or Poisson distribution that have a large amount of data and it can be seen that the polygons are getting closer to the normal curve when the amount of data is very large approaching infinity.

The writing of this article discusses the normal distribution approach through the binomial distribution and through the Poisson distribution. Further articles are expected to discuss more varied distribution approaches, given that there are many special probability distributions in statistics.

ACKNOWLEDGMENTS

The researchers would like to thank you for the opportunity to write this article, as well as all the people who have contributed to the preparation of this article. We also thank our supervisors, Ira Kurniawati, S.Si., M.Pd. and Dr. Triyanto, S.Si., M.Si. as well as our classmates. We hope that the readers can benefit from this article.

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