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CHROMATIC NUMBER OF AMALGAMATION OF WHEEL GRAPH-STAR GRAPH AND AMALGAMATION OF WHEEL GRAPH-SIKEL GRAPH

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Abstrak: Graf dinotasikan dengan G=(V,E) adalah pasangan himpunan(V,E) dimana V adalah himpunan tak kosong dari simpul-simpul pada G, sedangkan E adalah himpunan sisi pada G. Dalam teori graf dikenal berbagai jenis graf diantaranya graf bintang, graf sikel, dan graf roda. Operasi graf pada dua jenis graf atau lebih dapat menghasilkan graf baru. Amalgamasi adalah salah satu operasi pada graf. Misalkan G_1 dan G_2 dua graf terhubung, amalgamasi simpul dari graf G_1 dan graf G_2 dengan menggabungkan simpul v ∈ V(G_1) dan simpul u ∈ V(G_2) dinotasikan dengan amalt(G_1,G_2;v,u) adalah graf yang diperoleh dengan menggabungkan simpul v dari graf G_1 dan simpul u dari graf G_2 menjadi satu simpul x, dimana x adalah simpul bersama dari graf hasil $\text{amal}_t(G_1,G_2;u,v)$, sedangkan amalgamasi sisi dari graf G_1 dan graf G_2 dengan menggabungkan sisi $e \in E(G_1)$ dan sisi $f \in E(G_2)$, dinotasikan dengan $\text{amal}_s(G_1,G_2;e,f)$ adalah graf yang diperoleh dengan menggabungkan sisi e dari graf G_1 dan sisi f dari graf G_2 menjadi satu sisi g, dimana g adalah sisi bersama dari graf hasil $\text{amal}_s(G_1,G_2;e,f)$. Pewarnaan simpul merupakan salah satu topik yang berkembang pesat dalam graf. Pewarnaan simpul pada graf, yaitu pemberian warna pada setiap simpul graf sedemikian hingga setiap simpul yang bertetangga memiliki warna yang berbeda. Jumlah warna minimum yang digunakan untuk memberikan pewarnaan simpul pada suatu graf disebut dengan bilangan khromatik. Pada artikel ini dibahas mengenai bilangan khromatik pada graf hasil operasi amalgamasi graf roda dan graf bintang serta bilangan khromatik pada graf hasil operasi amalgamasi graf roda dan graf sikel. Hasil yang diperoleh adalah sebagai berikut: (1) bilangan khromatik graf hasil operasi amalgamasi dari graf roda (W_n) dan Graf bintang (S_m) untuk n ∈ genap adalah 3 dan untuk n ∈ ganjil adalah 4, (2) bilangan khromatik graf hasil operasi amalgamasi dari graf Roda (W_n) dan Graf Sikel (C_m) untuk n ∈ genap adalah 3 dan untuk n ∈ ganjil adalah 4.

Kata kunci : *Amalgamasi, Bilangan Khromatik, Graf Roda, Graf Bintang, dan Graf Sikel*

Abstract: A graph denoted by $G = (V, E)$ is a pair of $\text{set}(V, E)$ where V is a non-empty set of vertices in G, and E is a set of edges in G. In graph theory, there are various types of graphs including star graphs, cycle graph, and wheel graph. Graph operations on two or more types of graphs can produce new graphs. Amalgamation is one of the operations on graphs. Suppose G_1 and G_2 are two connected graphs, amalgamating the vertices of graph 1 and graph 2 by joining vertices $v \in V(G_1)$ and vertices $u \in V(G_2)$ denoted by $\text{amal}_t(G_1, G_2; v, u)$ is the graph obtained by joins vertex v of graph G_1 and vertex u of graph G_2 into one vertex x, where x is the common vertex of the graph resulting from $amal_t(G_1, G_2; u, v)$, while amalgamating the edges of graph G_1 and graph G_2 by joining edges $e \in E(G_1)$ and edge $f \in E(G_2)$, denoted by $\text{amal}_s(G1, G2; e, f)$

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are graphs obtained by combining edge of graph 1 and edge of graph 2 into one edge g, where g is common side of graph of $\text{amal}_s(G1,G2;e,f)$. Vertex coloring is one of the fastest-growing topics in graphs. The coloring of the vertices, that is, assigning a color to each vertex of the graph so that each neighboring vertex has a different color. The minimum number of colors used to color the vertices of a graph is called the chromatic number. In this article, we discuss the chromatic number in the amalgamated graph of wheel graph and star graph and the chromatic number on the amalgamation of wheel graph and cycle graph. The results obtained are as follows: (1) the chromatic number of the amalgamated graph of the wheel graph (W_n) and the star graph (S_m) for n even is 3 and for n odd is 4, (2) the chromatic number of the graph resulting from the operation amalgamation of the wheel graph (W_n) and the cycle graph (C_m) for n even is 3 and for n odd is 4.

Keywords: *Amalgamation, Chromatic Numbers, Wheel Graphs, Star Graphs, and Cycle Graphs*

INTRODUCTION

The graph denoted by $G = (V, E)$ is a pair of sets (V, E) where V is a non-empty set of vertexes in G, while E is a set of edges in G that connects a pair of vertices. The number of vertices of the graph G is denoted $|V(G)|$ is called an order graph, while the number of edges of G is denoted $|E(G)|$ called the size graph (M. Rajesh Kannan, 2022). There are many types of graphs, including star graphs, cycle graphs, and wheel graphs. Star graph (S_m) is a graph that has m vertices with one vertex of degree m-1 and the other vertices of degree one. A cycle graph is a simple graph in which every vertex has a degree of two. A cycle graph with m vertices is denoted by C_m. A wheel graph (W_n) is a graph composed of a cycle graph (C_m) and a complete graph (K_1), where each vertex of the cycle graph is directly connected to the complete graph K_1. The wheel graph W_n has n+1 vertices.

In graph theory, there are various types of graph operations that can produce new graphs. Amalgamation is one of the operations on graphs. Suppose $\{G_i | i \in \{1,2,3 \dots, m\}\}$ for $m \in N$ and $m \geq 2$, is a finite set of graphs and each G_i has a fixed vertex v_{0i} or a fixed edge e_{0i} which is called the terminal. A vertex amalgamation denoted by Amal (G_i, v_{0i}) is a graph formed by joining all G_i by concatenating the terminal vertices. While the edge amalgamation denoted by Amal (G_i, e_{0i}) is a graph formed by combining all G_i by joining the terminal edges (Yijun Xiong, 2019).

The amalgamation operation on a graph is known as a vertex amalgamation operation and an edge amalgamation operation. The graph that results from the amalgamation of vertex amalgamation of wheel graph and star graph $(W_n, S_m; v_1, u_2)$ is a graph obtained by combining vertices $u_1 \in V(W_n)$ and vertices $u_2 \in V(S_m)$ into one vertex x, where x is a vertex together from the graph of amalgamation operation results of $\text{amal}_t(W_n, S_m; v_1, u_2)$. The graph that results from the amalgamation of the wheel graph and the star graph $\text{amal}_s(W_n, S_m; v_1v_2, u_2u_1)$ is the graph obtained by combining edge $v_1\, v_2\, E(W_n)$ and edge $u_2\, u_1 \in E(S_m)$ into one edge ab, where ab is the common side of the graph resulting from operation $\mathrm{amal_s}(W_n, S_m; v_1\, v_2, u_2\, u_1)$. Meanwhile, the result of the amalgamation of wheel graph and cycle graph $\mathrm{amal_t}(W_n, C_m\,; v_1, u_1\,)$ is the graph obtained by combining vertex $u_1 \in V(W_n)$ and vertex $u_1 \in V(C_m)$ into one vertex x, where x is common vertices of the graph of amalgamation operation $\mathrm{amal_t}(W_n, C_m\,; v_1, u_1\,)$. The graph resulting from the amalgamation of the wheel graph and the cycle graph $\mathrm{amal_s}(W_n, C_m; v_1\, v_2, u_2\, u_1)$ is the graph obtained by combining edge $v_1\, v_2 \in E(W_n)$ and edge $u_2\, u_1 \in E(C_m)$ into one edge ab, where ab is the common side of the graph of amalgamation operation $\mathrm{amal_s}(W_n, C_m; v_1 v_2, u_2 u_1)$.

Graph coloring is one of the topics of discussion in graphs that is growing quite rapidly. Graph coloring is the mapping of colors to vertices, edges, or regions of the graph such that each neighboring vertex, edge, or region has a different color (Fangfang Wu, 2022). If n is the number of colors used to color graph G then the coloring is called n-coloring of G and if n is the minimum number of colors so that it has n-coloring then n is called the chromatic number of graph G (Yangyan Gu, 2022) . In vertex coloring, the minimum number of colors that can be used to color the vertices of a graph G is called the chromatic number of G, symbolized by $\chi(G)$. In developed studies, many properties of chromatic numbers in special graphs have been found, including chromatic numbers in empty graphs, chromatic numbers in complete graphs, chromatic numbers in bipartition graphs, chromatic numbers in star graphs, chromatic numbers in cycle graphs, and chromatic number in a wheel graph. In this study, we will investigate the chromatic number of the amalgamating operation of a wheel graph and star graph and the chromatic number of the graph of amalgamation of wheel and cycle graphs.

LITERATURE REVIEW

Definition 1:

The graph denoted by $G = (V, E)$ is a pair of sets (V, E) where V is a non-empty set of vertices (vertices) in G, while E is a set of edges in G that connects a pair of vertices.

Definition 2:

A complete graph is a graph in which every two distinct vertices are directly connected so that every vertex in a complete graph is related to each other. A complete graph is denoted by K_n, with $n \geq 1$.

Definition 3:

A graph G is said to be a bipartition graph if $V(G)$ can be partitioned into two mutually exclusive subsets X and Y, so that each edge in G has one end of a vertex in X and the other end is a vertex in Y.

Definition 4:

A cycle graph is a simple graph in which every vertex has a degree of two. A cycle graph with m vertices is denoted by C_m. The number of edges of a cycle graph that has m vertices is m.

Definition 5:

A star graph is a graph with m vertices where $m \geq 3$, with one vertex having degree m-1 and another vertex having degree one. A node with degree m-1 is called a central node and a node with degree 1 is called a terminal vertex. A star graph with m vertices is denoted by S_m.

Definition 6:

The wheels graph W_n is a graph that is the result of the addition of a C_n cycle graph with a complete graph K_1 or a graph obtained by adding a new vertex to the C_n cycle graph, so that each vertex in the C_n cycle graph is directly related to the new vertex. The new vertex connected to the node in the cycle is called the central vertex and the vertex in the cycle is called the terminal node. The number of vertices in the wheel graph W_n is n+1, while the number of edges is 2n.

Definition 7:

In general, the definition of amalgamation operation is as follows.

Let $\{G_i | i \in \{1,2,3 \ldots, m\}\}$ for $m \in N$ and $m \geq 2$, is a finite set of graphs and each G_i has a fixed vertex v_{0i} or a fixed edge e_{0i} which is called the terminal. A vertex amalgamation denoted by Amal (G_i, v_{0i}) is a graph formed by joining all G_i by concatenating the terminal vertices. Meanwhile, vertex amalgamation denoted by Amal (G_i, e_{0i}) is a graph formed by combining all G_i by joining the terminal edges (Yijun Xiong, 2019, p. 147345). Based on this, suppose 1 and 2 are two connected graphs, the amalgamation of the vertices of graph 1 and graph 2 by combining vertices $v \in V(G1)$ and vertices $u \in V(G2)$ denoted by $\mathrm{amal_t}(G1, G2; v, u)$ is the graph obtained by combining the vertices v of graph G_1 and vertices u of graph G_2 into one vertex x, where x is the common vertex of the amalt graph $(G1, G2\,; u, v)$. Meanwhile, the side amalgamation of graph G_1 and graph G_2 by combining edge $e \in E(G_1)$ and edge $f \in E(G_2)$, denoted by $\mathrm{amal_s}(G_1, G_2; e, f)$ is the graph obtained by combining edge of graph G_1 and edge of graph 2 becomes one edge , where is the common side of graph $\mathrm{amal_s}(G_1, G_2; e, f)$.

Definition 8:

Graph coloring is the mapping of colors to vertices, edges, or regions of the graph such that each neighboring vertex, edge, or region has a different color. If n is the number of colors used to color graph G then the coloring is called n-coloring of G and if n is the minimum number of colors so that it has n-coloring then n is called the chromatic number of graph G (Yangyan Gu, 2022) . In vertex coloring, the minimum number of colors that can be used to color the vertices of a graph G is called the chromatic number of G which is symbolized by (G)(Leonardo Mart´ınez, 2015).

Theorem 1

If there is a k-coloring in graph G, then $\chi\,(G) \leq k$

Proof:

If there is a k-coloring in graph G, then all vertices in G can be colored with k colors. Since the chromatic number is the minimum number of colors used to color all the vertices in graph G, such that the vertex coloring conditions are met, it is proved that $\chi\ (G)\ \leq\ k$

Theorem 2:

A complete graph of order n with $n \in N$, then $\chi(Kn)\ =\ n$.

Proof:

Because in a complete graph K_n every two vertices are connected by an edge so that every vertex in a complete graph will be interconnected, according to the definition of vertex coloring, all vertices must be colored with a different color. So the color needed to color the graph K_n is n, it proves that $\chi(Kn)\ =\ n$.

Theorem 3:

Let G be a non-empty graph, G is bipartition if and only if $\chi(G)\ =\ 2$

Proof:

(=>) If G is bipartition then $\chi(G)\ =\ 2$

If G is bipartitioned, then V(G) can be partitioned into 2 sets, let X and Y are such that X∩Y=ϕ and e(u,v)∈E(G) u∈X and v∈Y. From the understanding of the bipartition graph, u∈X can be given the same color (eg color 1) and u∈Y can be given the same color other than color 1 (eg color 2). Based on this, the minimum color to be able to color the vertices in a bipartition graph is 2 colors. then it is proven that (G) = 2.

(<=) If $\chi(G) = 2$ then G is bipartition

Take $v_i \in V(G)$ which has color 1. Assume that $v_i\ \in\ V(G)$ is a member of the set X. Take $u_i \in V(G)$ which has 2 colors. Assume that $ui \in V(G)$ is a member of the set Y. because $\chi(G) = 2$ then $V(G)$ is partitioned into X and Y with $X \cap Y = \phi$. Because the vertices at X and the vertices at Y are of different colors, according to the rule of coloring the sides in G connect the vertices at X and the vertices at Y (2 directly connected vertices have different colors). So it is proved that G is a graph where V(G) can be partitioned into two sets X and Y such that $X \cap Y = \phi$ and $e(u, v) \in E(G)\ u \in X$ and $v \in Y$ or G is a bipartition graph.

Theorem 4:

If C_n is a cycle graph with n vertices then $\chi(\mathrm{C_n}) = \begin{cases} 2, & \text{for n} \in \text{even} \\ 3, & \text{for n} \in \text{odd} \end{cases}$

Proof:

Let C_n be a cycle graph with n vertices, then the length of C_n is n.

The proof will be divided in 2 cases.

Case 1: if n is even, for example $C_n = (v_1\ v_2\ v_3\ v_4 \ldots v_n\ v_1)$ then $V(C_n)$ can be partitioned into 2 sets, namely $X = \{v_1, v_3, v_5, \ldots v_{n-1}\}$ and $Y = \{v_2, v_4, v_6, \ldots v_n\}$ where none of $\forall \mathrm{v_i} \in \mathrm{X}$ is directly connected and $\forall \mathrm{v_j} \in \mathrm{Y}$ is also not directly connected and $\forall \mathrm{e(u,v)} \in \mathrm{E(C_n)}$ $\mathrm{u} \in \mathrm{X}$ dan $\mathrm{v} \in \mathrm{Y}$. So it is proved that Cn for even n is a bipartition graph, so that $\chi(Cn) = 2$ for n even

Case 2: if n is odd then C_n is not a bipartition graph. Let $C_n = (v_1\ v_2\ v_3\ v_4 \ldots v_n\ v_1)$ take v_n from C_n. v_n is a node that is directly connected to v_1 and v_{n-1}.

The graph $C_n - v_n$ obtained from C_n by removing vertices vn is a bipartition graph where $V(C_n - v_n)$ can be partitioned into 2 sets $X = \{v_1, v_3, v_5, \ldots v_{n-2}\}$ and $Y = \{v_2, v_4, v_6, \ldots v_{n-1}\}$ and $X \cap Y = \phi$ and $\forall \mathrm{e(u,v)} \in \mathrm{E(C_n - v_n)}$, $\mathrm{u} \in \mathrm{X}$ dan $\mathrm{v} \in \mathrm{Y}$. So the graph $C_n - v_n$ can be colored with a minimum of 2 colors or $(C_n - v_n) = 2$. Because graph C_n can be formed from graph $C_n - v_n$ by adding back v_n where v_n is directly connected to $v_1 \in X$ and $v_{n-1} \in Y$, to color the vertices v_n on graph C_n must be colored differently with colors v_1 and v_{n-1}. So it is proved that $(C_n) = 2 + 1 = 3$ for $\mathrm{n} \in$ odd.

Theorem 5:

If W_n is a wheel graph with n vertices, then $\chi(\mathrm{W_n}) = \begin{cases} 3, \mathrm{n} \in \text{even} \\ 4, \mathrm{n} \in \text{odd} \end{cases}$

Proof:

Note that the wheel graph (W_n) is a graph composed of the sum of a cycle graph C_n and a complete graph K_1 where every vertex in the cycle is directly connected to the central vertex (complete graph (K_1)). So $\mathrm{W_n} = \mathrm{C_n} + \mathrm{K_1}$.

Based on Theorem 4 the chromatic number in the C_n cycle graph, is $\chi(\mathrm{C_n}) = \begin{cases} 2, \mathrm{n} \in \text{even} \\ 3, \mathrm{n} \in \text{odd} \end{cases}$

Based on Theorem 2 the chromatic number in the complete graph K_n is obtained $\chi(\mathrm{K_n}) = \mathrm{n}$, then for $\chi(\mathrm{K_1}) = 1$

Since $\mathrm{W_n} = \mathrm{C_n} + \mathrm{K_1}$ then $\chi(\mathrm{W_n}) = \chi(\mathrm{C_n}) + \chi(\mathrm{K_1})$. So that it is obtained

$$\chi(\mathrm{W_n}) = \begin{cases} 2 + 1, \mathrm{n} \in \text{even} \\ 3 + 1, \mathrm{n} \in \text{odd} \end{cases} \quad \text{or} \quad \chi(\mathrm{W_n}) = \begin{cases} 3, \mathrm{n} \in \text{even} \\ 4, \mathrm{n} \in \text{odd} \end{cases}$$

Theorem 6:

If S_m is a star graph for every integer m ≥ 3, then $\chi(S_m) = 2$

Proof:

Notice that the star graph (S_m) is a bipartition graph $K_{1,m-1}$. A star graph has one vertex of degree m-1 called the central vertex and m-1 vertices of degree one are called leaves. Since the star graph is a bipartition graph, based on Theorem 2 only 2 colors are needed to color the bipartition graph, so it is proven that $\chi(S_m) = 2$.

RESULT AND DISCUSSION

Wheel Graph and Star Graph Amalgamation Operation

The graph resulting from the amalgamation of the vertices of the wheel graph and the star graph is symbolized by $\mathrm{amal_t}(W_n, S_m; v_1, u_2)$ is the graph obtained by combining the vertices $u1 \in V(W_n)$ and the vertices $u_2 \in V(S_m)$ into one vertex x, where x is the common vertex of the graphs W_n and S_m

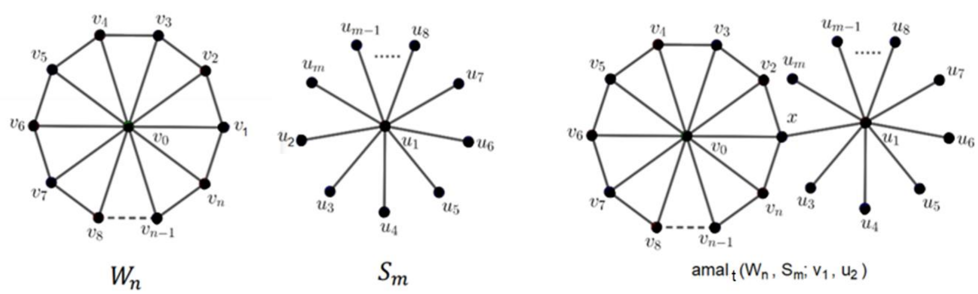


Figure 1. The Vertex Amalgamation Operation on Graph W_n and Graph S_m

The graph resulting from the amalgamation of the wheel graph and star graph is symbolized by $\mathrm{amal_s}(W_n, S_m; v_1 v_2, u_2 u_1)$ is the graph obtained by combining edges $v_1 v_2 \in E(W_n)$ and edges $u_2 u_1 \in E(S_m)$ into one edge ab, where ab is the common side of graph W_n and S_m.

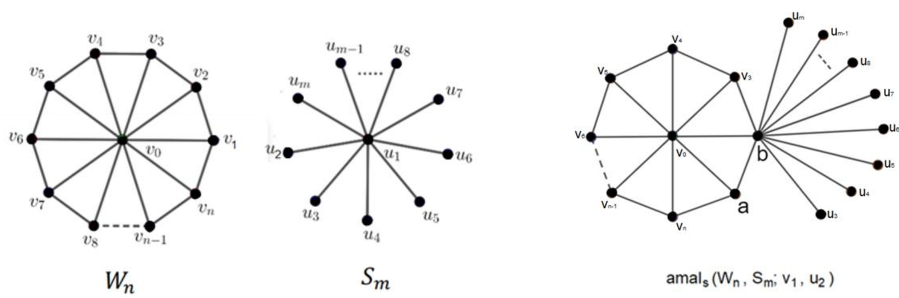


Figure 2. Edge Amalgamation Operation on Graphs W_n and S_m

Wheel and Cycle Graph Amalgamation Operations

The graph resulting from the amalgamation of vertices on wheel graphs and cycle graphs is symbolized by $\mathrm{amal_t}(W_n, C_m; v_1, u_1)$ is a graph obtained by combining vertices $u1 \in V(W_n)$ and vertices $u1 \in V(C_m)$ into one vertex x, where x is the common vertex of the graphs W_n and C_m.

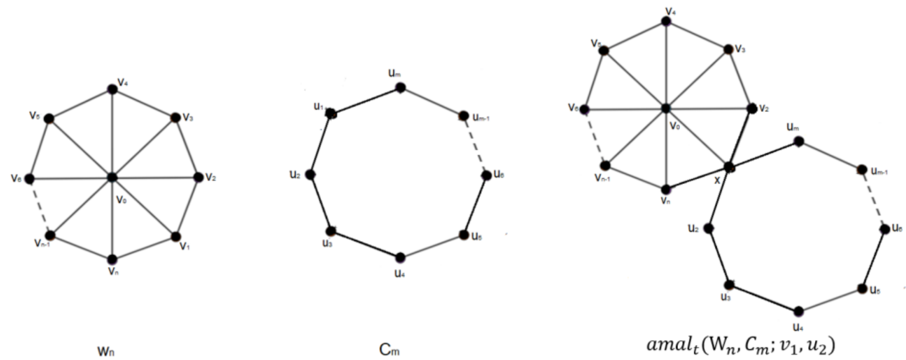


Figure 3. The Vertex Amalgamation Operation on a Araph W_n and graf C_m

The graph resulting from the amalgamation of the wheel graph and the cycle graph is symbolized by $\mathrm{amal_s}\ (\mathrm{W_n}, \mathrm{C_m}\ ; \mathrm{v_1 v_2}, \mathrm{u_2 u_1})$is the graph obtained by combining edge $v_1\, v_2 \in E(W_n)$ and edge $u_2\, u_1 \in E(C_m)$ into one edge ab, where ab is the common edge of graphs W_n and C_m.

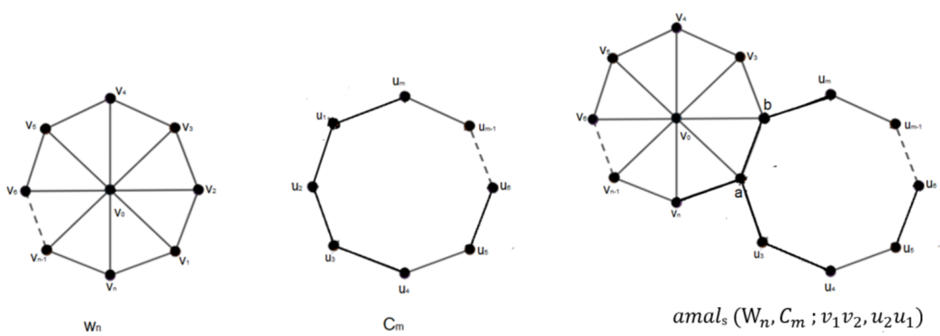


Figure 4. The Vertex Amalgamation Operation on A Graph W_n and graf C_m

Graph Coloring

Graph coloring is a mapping of colors on graph elements. Graph coloring is the mapping of colors to vertices, edges, or regions of the graph such that each neighboring vertex, edge, or region has a different color (Afriantini, Helmi, and F. Fran., 2019). If n is the number of colors used to color graph G, then the coloring is called n-coloring of G and if n is the minimum number of colors so that it has n-coloring then n is called the chromatic number of graph G.

Vertex Coloring in Amalgamated Graphs of Wheel Graphs and Star Graphs

Coloring of vertices in graph $\mathrm{amal_t}(W_n, S_m\ ; v_1, u_2)$ is Coloring of vertices in graph $\mathrm{amal_t}(W_n, S_m\ ; v_1, u_2)$ such that each neighboring vertex has a different color. The minimum number of colors that can be used to color the vertices of a graph $\mathrm{amal_t}(W_n, S_m\ ; v_1, u_2)$ is called the chromatic number of $\mathrm{amal_t}(W_n, S_m\,; v_1, u_2)$ which is symbolized by $\chi(\mathrm{amal_t}(W_n, S_m\,; v_1, u_2))$. The following is the coloring of the vertices in the graph $\mathrm{amal_t}(W_n, S_m\,; v_1, u_2)$ divided into 2 cases:

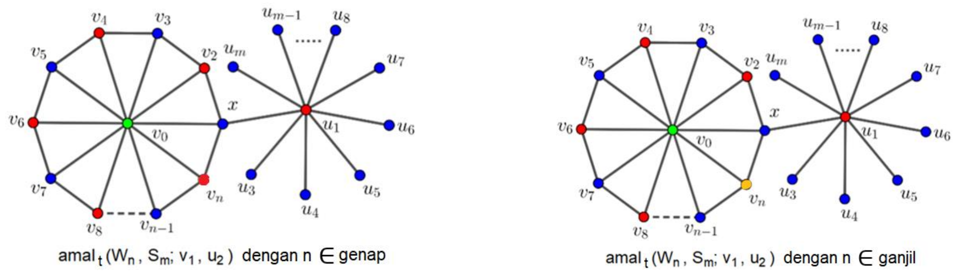


Figure 5. Coloring on Graph $\mathrm{Amal_t}(\mathrm{W_n}, \mathrm{S_m}\,; \mathrm{v_1}, \mathrm{u_2})$

Based on Figure 5, it can be seen that the minimum number of colors that can be used to color the vertex amalgamation graph of the wheel graph (W_n) and the star graph (S_m) is 3 for n ∈ even and 4 for n ∈ odd.

The vertex coloring in the graph $\mathrm{amal_s}(W_n, S_m\,; v_1\,v_2, u_2\,u_1)$ is the assignment of the vertex color to the graph $\mathrm{amal_s}(W_n, S_m\,; v_1\,v_2, u_2\,u_1)$ such that each neighboring vertex has a different color. The minimum number of colors that can be used to color the vertices of a graph $\mathrm{amal_s}(W_n, S_m\,; v_1\,v_2, u_2\,u_1)$ is called the chromatic number of $\mathrm{amal_s}(W_n, S_m\,; v_1\,v_2, u_2\,u_1)$ is symbolized by $\chi(\mathrm{amal_s}(W_n, S_m\,; v_1\,v_2, u_2\,u_1)))$. The following is the coloring of the vertices of the graph $\mathrm{amal_s}(W_n, S_m\,; v_1\,v_2, u_2\,u_1)$ divided into 2 cases:

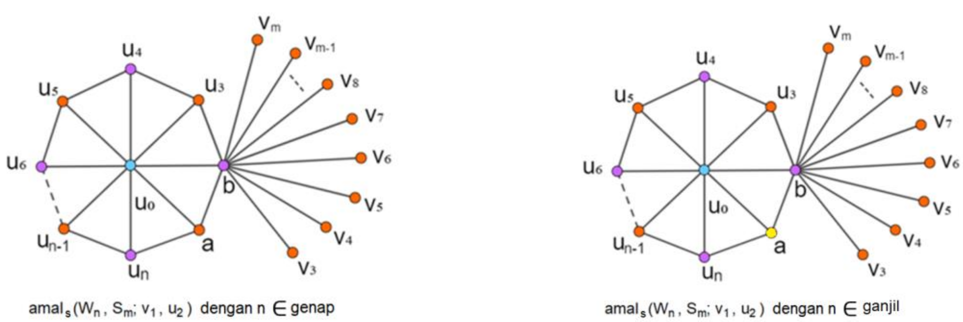


Figure 6. Coloring on Graph $\mathrm{Amal_s}(\mathrm{W_n}, \mathrm{S_m}\,; \mathrm{v_1 v_2}, \mathrm{u_2 u_1}$

Based on Figure 6, it can be seen that the minimum number of colors that can be used to perform coloring on the amalgamated edge graph of wheel graph (W_n) and star graph (S_m) is 3 for n even and 4 for n odd.

Based on these results, the following conclusions are obtained. If G is the result of the amalgamation operation of the wheel graph (W_n) and the star graph (S_m) then

$$\chi(G) = \chi(\mathrm{amal_t}(W_n, S_m\, ; v_1, u_2)) = \chi(\mathrm{amal_s}(W_n, S_m\, ; v_1\, v_2, u_2\, u_1)) = \begin{cases} 3, \mathrm{n} \in \text{even} \\ 4, \mathrm{n} \in \text{odd} \end{cases}$$

Proof:

1. Proof of $\chi\big(\mathrm{amal_t}(\mathrm{W_n}, \mathrm{S_m}\, ; \mathrm{v_1}, \mathrm{u_2})\big) = \begin{cases} 3, \mathrm{n} \in \text{even} \\ 4, \mathrm{n} \in \text{odd} \end{cases}$

   $\mathrm{amal_t}(W_n, S_m\, ; v_1, u_2)$ is an amalgamation of graph vertices $(W_n; v_1)$ and graph $(S_m; u_2)$ is an operation to form a new graph by combining two vertices, namely vertex $v_1 \in V(W_n)$ and vertex $u_2 \in V\,(S_m)$ into one vertex x. This vertex x is the common vertex between the wheel graph W_n and the star graph S_m. Because $(W_n) > \chi(S_m)$ the coloring of the graph resulting from the operation $\mathrm{amal_t}(W_n, S_m\, ; v_1, u_2)$ starts from coloring the wheel graph W_n first.

   a. According to Theorem 5, for n ∈ even, $\chi(\mathrm{W_n}) = 3$, so 3 colors are needed to color W_n. Furthermore, the remaining uncolored vertices from the graph of $\mathrm{amal_t}(W_n, S_m\, ; v_1, u_2)$ are m-1 vertices. From these m-1 nodes, there is 1 node of degree m-1 which is directly connected to node x (a joint vertex of W_n and S_m) and is directly connected to other m-2 nodes that have not been colored. m-2 this uncolored vertex is degree one. So, to color m-1 residual vertices, we can use two colors that have been used to color the vertices in W_n, m-2 vertices of degree one are colored with the same color as the color of vertex x, and take another color that has been used for coloring Wn to color a vertex of degree m-1. So, for n ∈ genap, $\chi(\mathrm{amal_t}\,(\mathrm{W_n}, \mathrm{S_m}\, ; \mathrm{v_1}, \mathrm{u_2})) = 3$

   b. According to theorem 5 for n ∈ ganjil, $\chi(\mathrm{W_n}) = 4$, so 4 colors are needed to color W_n. Furthermore, the remaining uncolored vertices from the graph of $\mathrm{amal_t}\,(\mathrm{W_n}, \mathrm{S_m}\, ; \mathrm{v_1}, \mathrm{u_2})$ are m-1 vertices. From these m-1 nodes, there is 1 node of degree m-1 which is directly connected to node x (a joint node of W_n and S_m) and is directly connected to other m-2 nodes that have not been colored. m-2 this uncolored vertex is degree one. So, to color m-1 residual vertices, we can use two colors that have been used to color the vertices in W_n, m-2 vertices of degree one are colored with the same color as the color of vertex x, and take another color that has been used for coloring Wn to color a vertex of degree m-1. So, for $n \in$ odd, $\chi(\mathrm{amal_t}\,(\mathrm{W_n}, \mathrm{S_m}\, ; \mathrm{v_1}, \mathrm{u_2})) = 4$

2. Proof of $\chi(\mathrm{amal_s}\,(\mathrm{W_n}, \mathrm{S_m}\, ; \mathrm{v_1 v_2}, \mathrm{u_2 u_1})) = \begin{cases} 3, \mathrm{n} \in \text{genap} \\ 4, \mathrm{n} \in \text{ganjil} \end{cases}$

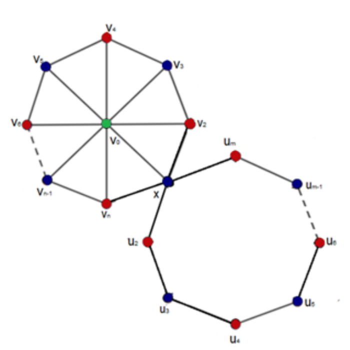
   $\mathrm{amal_s}\,(\mathrm{W_n}, \mathrm{S_m}\, ; \mathrm{v_1 v_2}, \mathrm{u_2 u_1})$ is an amalgamation of edges on a graph $(W_n\, ; v_1 v_2)$ and a graph $(S_m\, ;\, u_2 u_1)$ is an operation to form a new graph by combining two edges, namely edge $\mathrm{v_1 v_2} \in \mathrm{E(W_n)}$ and edge $\mathrm{u_2 u_1} \in \mathrm{E(S_m)}$ into one side ab. Edge ab is the common edge between the wheel graph W_n and the star graph S_m. Because $\chi(W_n) > \chi(S_m)$ coloring of the graph resulting from operation $amal_s(W_n, S_m ; v_1v_2, u_2u_1)$ starts with coloring the wheel graph W_n first.

a. According to theorem 5 for untuk $n \in$ even, $\chi(W_n) = 3$, so it takes 3 colors to color W_n. Furthermore, the remaining uncolored vertices from graph $amal_s(W_n, S_m ; v_1v_2, u_2u_1)$ are m-2 vertices which are vertices of degree 1, all of which are connected to symbol b in graph W_n. So, to color the remaining m-2 vertices, one color can be used that has been used to color the vertices in W_n (but must be different from the color in b). So, for $n \in$ even, $\chi(amal_s(W_n, S_m ; v_1v_2, u_2u_1)) = 3$

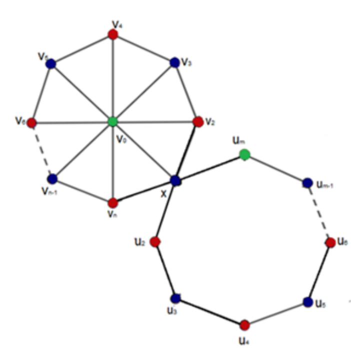
b. According to theorem 5 for $n \in$ odd, $\chi(W_n) = 4$, so 4 colors are needed to color W_n. Furthermore, the remaining uncolored vertices from graph $amal_s(W_n, S_m ; v_1v_2, u_2u_1)$ are m-2 vertices which are vertices of degree 1, all of which are connected to symbol b in graph W_n. So, to color the remaining m-2 vertices, one color can be used that has been used to color the vertices in W_n (but must be different from the color in b). So, for $n \in$ even, $\chi(amal_s(W_n, S_m ; v_1v_2, u_2u_1)) = 3$

Node Coloring of Graphs Resulting from Amalgamation of Wheel Graphs and Cycle Graphs

The coloring of the vertices in the graph $amal_t(W_n, C_m ; v_1, u_1)$ is the color assignment of the vertices in the graph $amal_t(W_n, C_m ; v_1, u_1)$ such that each neighboring vertex has a different color. The minimum number of colors that can be used to color the vertices of a graph $amal_t(W_n, C_m ; v_1, u_1)$is called the chromatic number of $amal_t(W_n, C_m ; v_1, u_1)$ symbolized by χ $(amal_t(W_n, C_m ; v_1, u_1)$. The following graph coloring $amal_t(W_n, C_m ; v_1, u_1)$ is divided into 4 cases:



$amal_t(W_n, C_m ; v_1, u_1)$ dengan n, m∈ genap



$amal_t(W_n, C_m ; v_1, u_1)$ dengan n∈ genap, m∈ ganjil

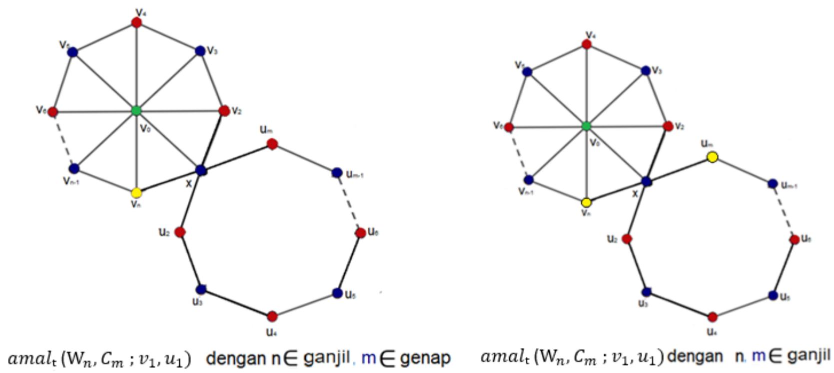


Figure 7. Coloring on Graph Amal$_t(W_n, C_m ; v_1, u_1)$

Based on Figure 7, it can be seen that the minimum number of colors that can be used to colorize the vertex amalgamation of wheel graph (W_n) and cycle graph (C_m) is 3 for n even and 4 for n odd..

The coloring of the vertices in the graph $amal_s(W_n, C_m ; v_1v_2, u_2u_1)$ is the assignment of the color of the vertices in the graph $amal_s(W_n, C_m ; v_1v_2, u_2u_1)$ such that each neighboring vertex has a different color. The minimum number of colors that can be used to color the vertices of a $amal_s(W_n, C_m ; v_1v_2, u_2u_1)$ is called the chromatic number of $amal_s(W_n, C_m ; v_1v_2, u_2u_1)$ is symbolized by $\chi(amal_s(W_n, C_m ; v_1v_2, u_2u_1))$. The following graph coloring $amal_s(W_n, C_m ; v_1v_2, u_2u_1)$ is divided into 4 cases:

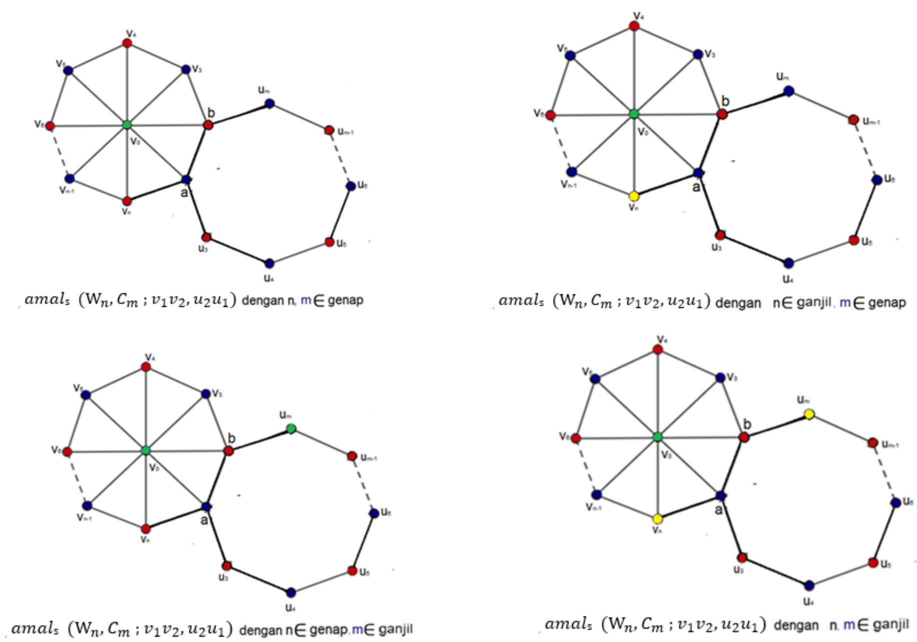


Figure 8. Coloring on Graph Amal$_s(W_n, C_m ; v_1v_2, u_2u_1)$

Based on Figure 8, it can be seen that the minimum number of colors that can be used to colorize the amalgamated edge graph of wheel graph (W_n) and cycle graph (C_m) is 3 for n even and 4 for n odd.

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Based on these results, the following conclusions are obtained.

If G is the result of amalgamation of wheel graph (W_n) and cycle graph (C_m) then

$$\chi(G) = \chi\left(\mathrm{amal_t}\ (\mathrm{W_n}, \mathrm{C_m}\ ; \mathrm{v_1}, \mathrm{u_1})\right) = \chi(\mathrm{amal_s}(\ \mathrm{W_n}, \mathrm{C_m}\ ; \mathrm{v_1 v_2}, \mathrm{u_2 u_1})) = \begin{cases} 3, \mathrm{n} \in \text{even} \\ 4, \mathrm{n} \in \text{odd} \end{cases}$$

Proof:

1. Proof of $\chi\left(\mathrm{amal_t}\ (\mathrm{W_n}, \mathrm{C_m}\ ; \mathrm{v_1}, \mathrm{u_1})\right) = \begin{cases} 3, \mathrm{n} \in \text{even} \\ 4, \mathrm{n} \in \text{odd} \end{cases}$

   $\mathrm{amal_t}\ (\mathrm{W_n}, \mathrm{C_m}\ ; \mathrm{v_1}, \mathrm{u_1})$ is an amalgamation of graph vertices ($W_n\ ; v_1$) and graph ($C_m\ ;\ u_1$) is an operation to form a new graph by combining two vertices, namely vertex $v_1 \in V(W_n)$ and vertex $u_1 \in V(C_m)$ into one vertex x. This vertex x is the common vertex between the wheel graph W_n and the cycle graph C_m. Because $\chi(\mathrm{W_n}) > \chi(\mathrm{C_m})$ the coloring of the graph of the result of operation $\mathrm{amal_t}(\mathrm{W_n}, \mathrm{C_m}\ ; \mathrm{v_1}, \mathrm{u_1})$ starts with the coloring of the wheel graph W_n first.

   a. According to theorem 5 for $\mathrm{n} \in$ even, $\chi(\mathrm{W_n}) = 3$, so 3 colors are needed to color W_n. Furthermore, the remaining uncolored vertices from the graph of $\mathrm{amal_t}(\mathrm{W_n}, \mathrm{C_m}\ ; \mathrm{v_1}, \mathrm{u_1})$ are m-1 vertices, namely the vertices in the graph $\mathrm{C_m} - \mathrm{u_1}$.

      1) Based on theorem 4, if m is even to color m-1 the vertices need 2 colors (using the color that has been used to color W_n).
      2) Based on theorem 4, then if m is odd to color m-1 the vertex requires 2 colors (using the color that has been used to color W_n, but must be different from the color at x is a vertex with W_n and C_m).

      Thus, for $\mathrm{n} \in$ even, $\chi(\mathrm{amal_t}\ (\mathrm{W_n}, \mathrm{C_m}\ ; \mathrm{v_1}, \mathrm{u_1})) = 3$

   b. According to Theorem 5, for $\mathrm{n} \in$ odd, $\chi(\mathrm{W_n}) = 4$, so 4 colors are needed to color W_n. Furthermore, the remaining uncolored vertices from the graph of $\mathrm{amal_t}\ (\mathrm{W_n}, \mathrm{C_m}\ ; \mathrm{v_1}, \mathrm{u_1})$ are m-1 vertices, namely the vertices in the graph $\mathrm{C_m} - \mathrm{u_1}$.

      1) Based on theorem 4, if m is even to color m-1 the vertices need 2 colors (using the color that has been used to color W_n).
      2) Based on theorem 4, if m is odd to color m-1 the vertices need 2 colors (using the color that has been used to color W_n, but must be different from the color in x which is the vertex with W_n and C_m).

      Thus, for $\mathrm{n} \in$ odd, $\chi(\mathrm{amal_t}\ (\mathrm{W_n}, \mathrm{C_m}\ ; \mathrm{v_1}, \mathrm{u_1})) = 4$

2. Proof of $\chi(\mathrm{amal_s}(W_n, C_m; v_1v_2, u_2u_1)) = \begin{cases} 3, n \in \text{even} \\ 4, n \in \text{odd} \end{cases}$

   The edge amalgamation of the wheel graph and the cycle $\mathrm{amal_s}(W_n, C_m; v_1v_2, u_2u_1)$ is a graph obtained by combining edge $v_1v_2 \in E(W_n)$ and edge $u_2u_1 \in E(C_m)$ into one edge ab, where ab is the common edge of the wheel graph W_n and the cycle graph C_m. Since $\chi(W_n) > \chi(C_m)$, the coloring of the graph resulting from the operation $\mathrm{amal_s}(W_n, C_m; v_1v_2, u_2u_1)$ starts with the coloring of the wheel graph W_n first.

   a. According to theorem 5 for $n \in$ even, $\chi(W_n) = 3$, so 3 colors are needed to color W_n. Furthermore, the remaining uncolored vertices from the graph of $\mathrm{amal_s}(W_n, C_m; v_1v_2, u_2u_1)$ are m-2 vertices, namely the vertices in the graph $C_m - u_2u_1$.
      1) Based on theorem 4, then if m is even to color the m-2 vertices it takes 2 colors (can use color to color vertices a and vertices b which are vertices with W_n and C_m).
      2) Based on theorem 4, then if m is odd to color the m-2 vertices it takes 3 colors (using the color that has been used to color W_n).

      Thus, for $n \in$ even, $\chi(\mathrm{amal_s}(W_n, C_m; v_1v_2, u_2u_1)) = 3$

   b. According to theorem 5 for $n \in$ odd, $\chi(W_n) = 4$, so 4 colors are needed to color W_n. Furthermore, the remaining uncolored vertices from the graph $\mathrm{amal_s}(W_n, C_m; v_1v_2, u_2u_1)$ are m-2 vertices, namely the vertices in the graph $C_m - u_2u_1$.
      1) Based on theorem 4, then if m is even to color the m-2 vertices it takes 2 colors (can use the color used to color vertices a and vertices b which are vertices with W_n and C_m).
      2) Based on theorem 4, then if m is odd to color the m-2 vertices it takes 3 colors((using the color that has been used to color W_n).

      Thus, for $n \in$ odd, $\chi(\mathrm{amal_s}(W_n, C_m; v_1v_2, u_2u_1)) = 4$

CONCLUSION

1. The graph $\mathrm{amal_t}(W_n, S_m; v_1, u_2)$ is the result of the vertex amalgamation operation of the wheel graph and star graph. This graph is formed by combining the vertices $u_1 \in V(W_n)$ and the vertices $u_2 \in V(S_m)$ into one vertex x, where x is the common vertex of the graphs W_n and S_m while the graph $\mathrm{amal_s}(W_n, S_m; v_1v_2, u_2u_1)$ is Graph of operating results. Side amalgamation of wheel and star graphs. This graph is formed by combining edges $v_1v_2 \in E(Wn)$ and edges $u_2u_1 \in E(Sm)$ into one edge ab, where ab is the common edge of graphs W_n and S_m. If G is the result of the amalgamation operation of the wheel graph (W_n) and the star graph (S_m) then

$$\chi(G) = \chi(\mathrm{amal_t}(W_n, S_m; v_1, u_2)) = \chi(\mathrm{amal_s}(W_n, S_m; v_1v_2, u_2u_1)) = \begin{cases} 3, n \in \text{even} \\ 4, n \in \text{odd} \end{cases}$$

2. The graph $\mathrm{amal_t}\,(\mathrm{W_n}, \mathrm{C_m}\,; \mathrm{v_1}, \mathrm{u_1})$ is the result of the vertex amalgamation operation of the wheel graph and the cycle graph. This graph is formed by combining vertices $v\,1\ \in\ V(Wn)$ and vertices $u_1 \in V(C_m)$ into one vertex x, where x is the common vertex of graphs W_n and C_m while graph $\mathrm{amal_s}(\mathrm{W_n}, \mathrm{C_m}\,; \mathrm{v_1 v_2}, \mathrm{u_2 u_1})$ is the result of an edge amalgamation operation of a wheel graph and a cycle graph. This graph is formed by combining edges $v_1 v_2\ \in\ E(Wn)$ and edges $u_2 u_1 \in E(C_m)$ into one edge ab, where ab is the common edge of graphs W_n and C_m. If G is the result of the amalgamation operation of the wheel graph (W_n) and the cycle graph (C_m) then

$$\chi(\mathrm{G}) = \chi\big(\mathrm{amal_t}\,(\mathrm{W_n}, \mathrm{C_m}\,; \mathrm{v_1}, \mathrm{u_1})\big) = \chi(\mathrm{amal_s}(\,\mathrm{W_n}, \mathrm{C_m}\,; \mathrm{v_1 v_2}, \mathrm{u_2 u_1})) = \begin{cases} 3, \mathrm{n} \in \mathrm{even} \\ 4, \mathrm{n} \in \mathrm{odd} \end{cases}$$

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