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New Mathematical Properties of the Kumaraswamy Lindley Distribution

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Abstract. The Kumaraswamy Lindley distribution is a generalized distribution that has many applications in various fields, including physics, engineering, and chemistry. This paper introduces new mathematical properties for Kumaraswamy Lindley distribution such as probability weighted moments, moments of residual life, mean of residual life, reversed residual life, cumulative hazard rate function, and mean deviation.

Keywords: hazard rate; Kumaraswamy Lindley distribution; mean deviation; probability weighted moments; residual life

1. INTRODUCTION

There are many researchers dealt with this type of similar distributions to the proposed Kumrsawamy Lindely distribution (KLD). Gauss, *et. al* [1] have proposed a new family of generalized distributions, Elbatal *et. al* [2] presented a new generalized Lindley distribution, and Çakmakyapan *et. al* [3] proposed a new customer lifetime duration distribution for the Kumaraswamy Lindley distribution. Oluyede *et. al* [4] devoted a generalized class of Kumaraswamy Lindley distribution with applications to lifetime data. Riad *et. al* [5] analyzed a log-beta log-logistic regression model. Cordeiro *et. al* [6] presented the Kumaraswamy normal linear regression model with applications. Cakmakyapan, *et. al* [3] presented the Kumaraswamy Marshall-Olkin log-logistic distribution with application. Vigas *et. al* [7] presented the Poisson-Weibull regression model. Nofal *et. al* [8] presented the transmuted Geometric-Weibull distribution and its regression model, and Rocha *et. al* [9] presented a negative binomial Kumaraswamy-G cure rate regression model. Handique *et. al* [10] presented Marshall-Olkin-Kumaraswamy-G family of distributions.

Eissa [11] presented exponentiated Kumaraswamy-Weibull distribution with application to real data and Elgarhy [12] proposed Kumaraswamy Sushila distribution. Altun *et. al* [13] presented a new generalization of generalized half-normal distribution. Abed *et. al* [14] proposed a new mixture statistical distribution exponential-Kumaraswamy. Fachini-Gomes, *et. al* [15] presented the bivariate Kumaraswamy Weibull regression model. Arshad, *et. al* [16] presented the gamma Kumaraswamy-G family distribution, theory, inference and applications. Mdlongwaa, *et. al* [17] presented Kumaraswamy log-logistic Weibull distribution, model theory and

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application to lifetime and survival data. Pumi *et. al* [18] presented Kumaraswamy regression model with Aranda-Ordaz link function. Safari *et. al* [19] presented Robust reliability estimation for Lindley distribution, a probability integral transform statistical approach, while Hafez *et. al* [20] presented a study on Lindley distribution accelerated life tests, application, and numerical simulation.

2. LINDLEY DISTRIBUTION AND KUMARASWAMY LINDLEY DISTRIBUTION

The Lindley distribution was introduced in 1958, but it was used as an alternative to the exponential distribution, where the Lindley distribution was used to study many characteristics such as data modeling and other characteristics. In this section, the definition and properties of Lindley distribution are provided. Equation (1) presents the probability distribution function (pdf) of the Lindley distribution with parameter θ:

$$g\,(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \quad x > 0\,, \theta > 0. \tag{1}$$

and the corresponding cumulative distribution function (cdf) is

$$G\,(x) = 1 - e^{-\theta x}\left(1 + \frac{x\,\theta}{\theta+1}\right) \; x > 0\,, \theta > 0. \tag{2}$$

Suppose $G(x,\theta)$ be the cdf of the Lindley distribution given by (2). The cdf and pdf of Kumaraswamy Lindley distribution (KLD) are

$$F\,(x) = 1 - \left[1 - \left(1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta+1}\right)\right)^a\right]^b. \tag{3}$$

$$f\,(x) = ab\left(\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\right)\left(1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta+1}\right)\right)^{a-1} \left[1 - \left(1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta+1}\right)\right)^a\right]^{b-1}. \tag{4}$$

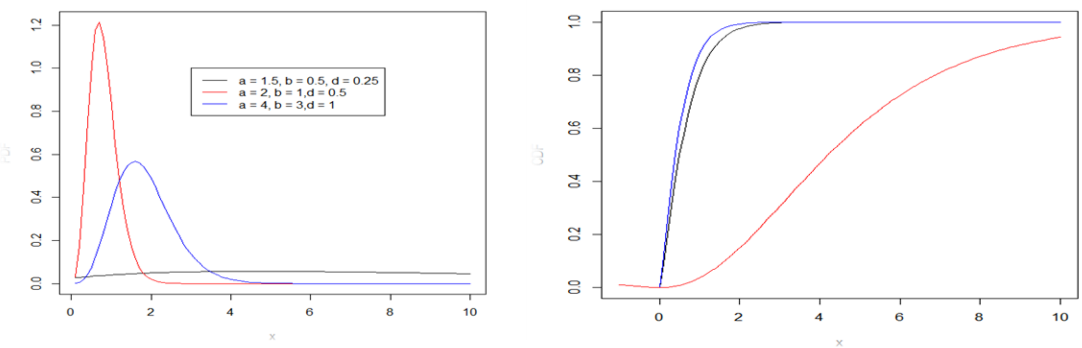

Figure 1. Plot of pdf (left) and cdf (right) of KLD

3. MATHEMATICAL PROPERTIES

3.1. Probability Weighted Moments

The probabilities weighted moments (PWM) are used to derive parameters of generalized probability distributions and this method is used to compare the parameters obtained by using the probabilities weighted moments:

$$\rho_{s,r} = E(x^s F(x)^r) = \int_0^\infty x^s\, F(x)^r f(x)dx. \tag{5}$$

From the definition of the cumulative distribution function of the Kumaraswamy Lindley, we find that

$$F\,(x) = 1 - \left[1 - \left(1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta + 1}\right)\right)^a\right]^b$$

$$(1 - z)^d = \sum_{i=0}^{\infty}(-1)^i \binom{d}{i} z^i, \qquad |z| < 1$$

$$\left(1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta + 1}\right)\right)^a = \sum_{n=0}^{\infty}(-1)^n \binom{a}{n} e^{-n\theta x}\left(1 + \frac{\theta x}{\theta + 1}\right)^n.$$

Using the binomial theorem on the following

$$\left(1 + \frac{\theta x}{\theta + 1}\right)^n = \sum_{l=0}^{\infty}\binom{n}{l}\left(\frac{\theta}{\theta + 1}\right)^l x^l$$

and substituting the value of the term into equation (3), we get

$$\left(1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta + 1}\right)\right)^a = \sum_{n=0}^{\infty}(-1)^n \binom{a}{n}\sum_{l=0}^{\infty}\binom{n}{l}\left(\frac{\theta}{\theta + 1}\right)^l x^l e^{-n\theta x}.$$

Simplifying the equation number, we get

$$\left(1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta + 1}\right)\right)^a = W_{nl}x^l e^{-n\theta x}$$

where

$$W_{nl} = \sum_{n=0}^{\infty}(-1)^n \binom{a}{n}\sum_{l=0}^{\infty}\binom{n}{l}\left(\frac{\theta}{\theta + 1}\right)^l.$$

Substituting into the cumulative distribution function for a value

$$\left(1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta + 1}\right)\right)^a$$

then

$$F\,(x) = 1 - \left[1 - W_{nl}x^l e^{-n\theta x}\right]^b.$$

Applying the binomial theorem to the following expression

$$\left[1 - W_{nl}x^l e^{-n\theta x}\right]^b = \sum_{\delta=0}^{\infty}(-1)^\delta \binom{b}{\delta}\left(W_{nl}x^l e^{-n\theta x}\right)^\delta$$

and simplifying the previous expansion, we get

$$\left[1 - W_{nl}x^l e^{-n\theta x}\right]^b = \sum_{\delta=0}^{\infty}(-1)^\delta \binom{b}{\delta}\,(W_{nl})^\delta \quad e^{-\delta n\theta x}x^{\delta l}$$

$$\left[1 - W_{nl}x^l e^{-n\theta x}\right]^b = W_{nl\delta}e^{-\delta n\theta x}x^{\delta l}$$

where

$$W_{nl\delta} = \sum_{\delta=0}^{\infty}(-1)^\delta \binom{b}{\delta}\,(W_{nl})^\delta.$$

Thus, the final formulation of the cumulative distribution function of the Kumaraswamy Lindley distribution becomes

$$F\,(x) = 1 - W_{nl\delta}e^{-\delta n\theta x}x^{\delta l}.$$

Now, we show the components of the general formulation the probabilities weighted moments (PWM). The first component is

$$F(x)^r = \left(1 - W_{nl\delta}e^{-\delta n\theta x}x^{\delta l}\right)^r.$$

By applying the binomial theorem to the following expansion

$$F(x)^r = \sum_{\pi=0}^{\infty}(-1)^{\pi}\binom{r}{\pi}\left(e^{-\pi\delta n\theta x}x^{\pi\delta l}\right)(W_{nl\delta})^{\pi}$$

then the final form of the function $F(x)^r$ is

$$F(x)^r = \left(e^{-\pi\delta n\theta x}x^{\pi\delta l}\right)W_{nl\delta\pi}$$

where $W_{nl\delta\pi} = \sum_{\pi=0}^{\infty}(-1)^{\pi}\binom{r}{\pi}(W_{nl\delta})^{\pi}$.

The second component is $f(x) = \frac{ab\,\theta^2}{\theta+1} w_j\, q\, x^{m+j}(1+x)\, e^{-\theta x(k+\gamma)}$. Substituting in the general form for probability weighted moments, we get

$$\rho_{s,r} = \int_0^{\infty} x^s\left(e^{-\pi\delta n\theta x}x^{\pi\delta l}\right)W_{nl\delta\pi}\frac{ab\,\theta^2}{\theta+1} w_j\, q\, x^{m+j}(1+x)e^{-\theta x(k+\gamma)}dx.$$

Simplifying the equation, we get

$$\rho_{s,r} = \frac{ab\,\theta^2}{\theta+1} w_j\, qW_{nl\delta\pi}\int_0^{\infty} e^{-\pi\delta n\theta x}\left(x^{\pi\delta l}\right)x^{s+m+j}(1+x)\, e^{-\theta x(k+\gamma)}dx$$

$$\rho_{s,r} = \frac{ab\,\theta^2}{\theta+1} w_j\, qW_{nl\delta\pi}\int_0^{\infty} x^{\pi\delta l+s+m+j}\ (1+x)\, e^{-\theta x(k+\gamma)-\pi\delta n\theta x}dx$$

$$\rho_{s,r} = \frac{ab\,\theta^2}{\theta+1} w_j\, qW_{nl\delta\pi}\int_0^{\infty}\left(x^{\pi\delta l+s+m+j+1} + x^{\pi\delta l+s+m+j}\right)e^{-\theta x(k+\gamma)-\pi\delta n\theta x}dx$$

$$\rho_{s,r} = \frac{ab\,\theta^2}{\theta+1} w_j\, qW_{nl\delta\pi}\int_0^{\infty}\left(x^{\pi\delta l+s+m+j+1} + x^{\pi\delta l+s+m+j}\right)e^{-(\theta(k+\gamma)+\pi\delta n\theta)x}dx$$

$$\rho_{s,r} = \tau\int_0^{\infty}\left(x^{\pi\delta l+s+m+j+1}\right)e^{-(\theta(k+\gamma)+\pi\delta n\theta)x}\,dx \quad + \tau\int_0^{\infty} e^{-(\theta(k+\gamma)+\pi\delta n\theta)x}\, x^{\pi\delta l+s+m+j}dx\,.$$

We assume that

$K_1 = \tau\int_0^{\infty}\left(x^{\pi\delta l+s+m+j+1}\right)e^{-(\theta(k+\gamma)+\pi\delta n\theta)x}\,dx$ and $K_2 = \tau\int_0^{\infty} e^{-(\theta(k+\gamma)+\pi\delta n\theta)x}\, x^{\pi\delta l+s+m+j}dx$.

It becomes probabilities weighted moments (PWM) as following $\rho_{s,r} = K_1 + K_2$ and we calculate the value of each K_1 and K_2:

$$K_1 = \tau\int_0^{\infty}\left(x^{\pi\delta l+s+m+j+1}\right)e^{-(\theta(k+\gamma)+\pi\delta n\theta)x}\,dx.$$

by assuming that $y = (\theta(k+\gamma)+\pi\delta n\theta)x$. Differentiating the hypothesis, we get

$$dy = (\theta(k+\gamma)+\pi\delta n\theta),\ x = \frac{y}{(\theta(k+\gamma)+\pi\delta n\theta)}\ ,\ dx = \frac{dy}{(\theta(k+\gamma)+\pi\delta n\theta)}.$$

By substituting the value of each of x, dx and simplifying it, we get

$$K_1 = \tau\int_0^{\infty}\left(\left(\frac{y}{(\theta(k+\gamma)+\pi\delta n\theta)}\right)^{\pi\delta l+s+m+j+1}\right)e^{-y}\ \frac{dy}{(\theta(k+\gamma)+\pi\delta n\theta)}$$

$$K_1 = \frac{\tau}{(\theta(k+\gamma)+\pi\delta n\theta)^{\pi\delta l+s+m+j+2}}\int_0^{\infty} y^{\pi\delta l+s+m+j+1}\ e^{-y}\ dy.$$

From the gamma integral we have

$$K_1 = \frac{\tau}{(\theta(k+\gamma)+\pi\delta n\theta)^{\pi\delta l+s+m+j+2}}\boldsymbol{\Gamma}\,(\pi\delta l+s+m+j+2)$$

and by calculating the value of K_2 we get

$$K_2 = \tau\int_0^{\infty} e^{-(\theta(k+\gamma)+\pi\delta n\theta)x}\, x^{\pi\delta l+s+m+j}dx.$$

Substituting in K_2 for the hypothesis and differentiation, we get

$$K_2 = \tau\int_0^{\infty} e^{-(y)}\left(\frac{y}{(\theta(k+\gamma)+\pi\delta n\theta)}\right)^{\pi\delta l+s+m+j}\frac{dy}{(\theta(k+\gamma)+\pi\delta n\theta)}.$$

Simplifying the previous equation, we have

$$K_2 = \frac{\tau}{(\theta(k+\gamma)+\pi\delta n\theta)^{\pi\delta l+s+m+j+1}}\int_0^{\infty} e^{-(y)}\,(y)^{\pi\delta l+s+m+j}dy.$$

From the gamma integral we get

$$K_2 = \frac{\tau}{(\theta(k+\gamma)+\pi\delta n\theta)^{\pi\delta l+s+m+j+2}} \boldsymbol{\Gamma}\,(\pi\delta l + s + m + j + 1).$$

The value of probabilities weighted moments (PWM) is $\rho_{s,r} = K_1 + K_2$,

$$\rho_{s,r} = \frac{\tau\boldsymbol{\Gamma}\,(\pi\delta l+s+m+j+2)}{(\theta(k+\gamma)+\pi\delta n\theta)^{\pi\delta l+s+m+j+2}} + \frac{\tau\,\boldsymbol{\Gamma}\,(\pi\delta l+s+m+j+1)}{(\theta(k+\gamma)+\pi\delta n\theta)^{\pi\delta l+s+m+j+2}}.$$

The *n*th moments of residual life denoted by $E[(x-t)^n |\, x > t]$ where $n = 1,2,3, \ldots.$ It is defined by

$$m(t)_n = \frac{1}{R(T)}\int_t^\infty (x-t)^n\, f(x)dx.$$

We substitute in the general form for the residual life for the probability density function of the Kumaraswamy Lindley distribution

$$m(t)_n = \frac{1}{R(T)}\int_t^\infty (x-t)^n\ \frac{ab\,\theta^2}{\theta\ +\ 1}\ w_j\ q\ x^{m+j}\ (1+x)e^{-\theta x(k+\gamma)}dx.$$

From the binomial theorem we have

$$(x-t)^n = (-t)^n(1-\frac{x}{t})^n = (-t)^n\sum_{h=0}^{\infty}(-1)^h\binom{n}{h}\left(\frac{x}{t}\right)^h.$$

Using the binomial expansion of the expression $((x-t)^n\ )$ and the substitution of the general form for residual life, we get

$$m(t)_n = \frac{1}{R(T)}\int_t^\infty (-t)^n\sum_{h=0}^{\infty}(-1)^h\binom{n}{h}\left(\frac{x}{t}\right)^h\ \frac{ab\,\theta^2}{\theta\ +\ 1}\ w_j\ q\ x^{m+j}\ (1+x)e^{-\theta x(k+\gamma)}dx.$$

By simplifying it we get

$$m(t)_n = \frac{\frac{ab\,\theta^2}{\theta\ +\ 1}\ w_j\ q(-t)^n\sum_{h=0}^{\infty}(-1)^h\binom{n}{h}}{R(T)}\int_t^\infty (\frac{x}{t})^h\ x^{m+j}\ (1+x)\ e^{-\theta x(k+\gamma)}dx.$$

Let $k_3 = \dfrac{\frac{ab\,\theta^2}{\theta\ +\ 1}\ w_j\ q(-t)^n\sum_{h=0}^{\infty}(-1)^h\binom{n}{h}}{t^h}$ then the equation becomes

$$m(t)_n = \frac{k_3}{R(T)}\int_t^\infty\ x^{m+j+h}\ (1+x)\ e^{-\theta x(k+\gamma)}dx$$

$$m(t)_n = \frac{k_3}{R(T)}\left[\int_t^\infty\ x^{m+j+h}\ e^{-\theta x(k+\gamma)}dx + \int_t^\infty x^{m+j+h+1}\ e^{-\theta x(k+\gamma)}dx\right].$$

Let $m(t)_n = \frac{k_3}{R(T)}[k_4 + k_5]$ where

$$k_4 = \int_t^\infty\ x^{m+j+h}\ e^{-\theta x(k+\gamma)}dx \text{ and } k_5 = \int_t^\infty\ x^{m+j+h+1}\ e^{-\theta x(k+\gamma)}dx.$$

Now, calculating the value of k_4 and using the general form of the gamma function $\boldsymbol{\Gamma}(a,b) = \int_b^\infty y^{a-1}e^{-y}\,dy$ we have

$$k_4 = \int_t^\infty\ x^{m+j+h}\ e^{-\theta x(k+\gamma)}dx.$$

We assume that $y = (\theta(k+\gamma))x$ and differentiate the hypothesis, then we get

$$dy = (\theta(k+\gamma))dx\ ,\ x = \frac{y}{(\theta(k+\gamma))}\ \ , dx = \frac{dy}{(\theta(k+\gamma))}.$$

By substituting the value of each of x, dx and simplifying it, we get

$$k_4 = \int_t^\infty\ \left[\frac{y}{(\theta(k+\gamma))}\right]^{m+j+h}\ e^{-y}\frac{dy}{(\theta(k+\gamma))}$$

$$k_4 = \frac{1}{(\theta(k+\gamma))^{m+j+h+1}}\int_{t\theta(k+\gamma)}^\infty\ [y]^{m+j+h}\ e^{-y}\ dy.$$

Applying the general form of the gamma function $-1 = m + j + h, b = t\theta(k+\gamma)$, then the value of k_4 is

$$k_4 = \frac{\Gamma(m+j+h+1, t\theta(k+\gamma))}{(\theta(k+\gamma))^{m+j+h+1}}.$$

To compute $k_5 = \int_t^\infty x^{m+j+h} e^{-\theta x(k+\gamma)} dx$, we assume that $y = (\theta(k+\gamma))x$. Differentiating the hypothesis, we get $dy = (\theta(k+\gamma))dx$, $x = \frac{y}{(\theta(k+\gamma))}$, $dx = \frac{dy}{(\theta(k+\gamma))}$. By substituting the value of each of x, dx in k_5 and simplifying it, we get

$$k_5 = \int_{\theta(k+\gamma)t}^{\infty} \left[\frac{y}{(\theta(k+\gamma))}\right]^{m+j+h+1} e^{-y} \frac{dy}{(\theta(k+\gamma))}$$

$$k_5 = \frac{1}{[\theta(k+\gamma)]^{m+j+h+2}} \int_{(\theta(k+\gamma))t}^{\infty} [y]^{m+j+h+1} e^{-y} dy.$$

Applying the general form of the gamma function $a - 1 = m + j + h + 1, b = t\theta(k+\gamma)$, then

$$k_5 = \frac{\Gamma(m+j+h+2, t\theta(k+\gamma))}{[\theta(k+\gamma)]^{m+j+h+2}}.$$

Moment of residual life is

$$m(t)_n = \frac{k_3}{R(T)}\left[\frac{\Gamma(m+j+h+1, t\theta(k+\gamma))}{(\theta(k+\gamma))^{m+j+h+1}} + \frac{\Gamma(m+j+h+2, t\theta(k+\gamma))}{[\theta(k+\gamma)]^{m+j+h+2}}\right].$$

3.2. Mean of Residual Life

The mean residual of KLD is defined by

$$m(t) = \frac{1}{R(T)}\int_t^\infty x f(x) dx - t.$$

Substituting the probability density function for the KLD, we get

$$m(t) = \frac{1}{R(T)}\int_t^\infty x \frac{ab\,\theta^2}{\theta + 1} w_j\, q\, x^{m+j} (1+x) e^{-\theta x(k+\gamma)} dx - t$$

Simplifying the equation, we have

$$m(t) = \frac{\frac{ab\,\theta^2}{\theta + 1} w_j\, q}{R(T)}\int_t^\infty x^{m+j+1} (1+x) e^{-\theta x(k+\gamma)} dx - t$$

$$m(t) = \frac{\frac{ab\,\theta^2}{\theta+1} w_j\, q}{R(T)}\left[\int_t^\infty x^{m+j+1} e^{-\theta x(k+\gamma)} dx + \int_t^\infty x^{m+j+2} e^{-\theta x(k+\gamma)}) \; dx\right] - t.$$

Let $k_6 = \int_t^\infty x^{m+j+1} e^{-\theta x(k+\gamma)} dx$, $k_7 = \int_t^\infty x^{m+j+2} e^{-\theta x(k+\gamma)})\; dx$ and $\delta = \frac{\frac{ab\,\theta^2}{\theta+1} w_j\, q}{R(T)}$, the mean of residual life becomes

$$m(t) = \delta[\, k_6 + \, k_7] - t.$$

Now, we calculate the value of k_6,

$$k_6 = \int_t^\infty x^{m+j+1} e^{-\theta x(k+\gamma)} dx$$

We assume that $y = (\theta(k+\gamma))x$. Differentiating the hypothesis, we get $dy = (\theta(k+\gamma))dx$, $x = \frac{y}{(\theta(k+\gamma))}$, $dx = \frac{dy}{(\theta(k+\gamma))}$. By substituting the value of each x, dx in k_6 and simplifying it, we get

$$k_6 = \int_t^\infty \left[\frac{y}{(\theta(k+\gamma))}\right]^{m+j+1} e^{-y} \frac{dy}{(\theta(k+\gamma))}$$

$$k_6 = \frac{1}{[(\theta(k+\gamma))]^{m+j+2}} \int_{(\theta(k+\gamma))t}^{\infty} [y]^{m+j+1} e^{-y} dy \; .$$

Applying the general form of the gamma function $a-1=m+j+1, b=(\theta(k+\gamma))t$, we have

$$k_6=\frac{\boldsymbol{\Gamma}(m+j+2,\big(\theta(k+\gamma)\big)t)}{[(\theta(k+\gamma))]^{m+j+2}}$$

Now, we calculate the value of k_7

$$k_7=\int_t^{\infty} x^{m+j+2}e^{-\theta x(k+\gamma)})\ dx.$$

We assume that $y=(\theta(k+\gamma))x$. Differentiating the hypothesis, we get $dy=(\theta(k+\gamma))dx$, $x=\frac{y}{(\theta(k+\gamma))}$, $dx=\frac{dy}{(\theta(k+\gamma))}$. By substituting the value of each x, dx in k_7, we get

$$k_7=\frac{1}{(\theta(k+\gamma))^{m+j+3}}\int_{(\theta(k+\gamma))t}^{\infty} y^{m+j+2}e^{-y})\ dy.$$

By applying the general form of the gamma function $a-1=m+j+2, b=(\theta(k+\gamma))t$, we have

$$k_7=\frac{\boldsymbol{\Gamma}(m+j+3,\big(\theta(k+\gamma)\big)t)}{(\theta(k+\gamma))^{m+j+3}}.$$

The mean of residual life becomes

$$m(t)=\delta\left[\frac{\boldsymbol{\Gamma}(m+j+2,\big(\theta(k+\gamma)\big)t)}{[(\theta(k+\gamma))]^{m+j+2}}+\frac{\boldsymbol{\Gamma}(m+j+3,\big(\theta(k+\gamma)\big)t)}{(\theta(k+\gamma))^{m+j+3}}\right]-t.$$

By differentiating the equation *m(t)* with respect to *x*, we have

$$\acute{m}(t)=\acute{\delta}\left[\frac{\boldsymbol{\Gamma}\big(m+j+2,\big(\theta(k+\gamma)\big)t\big)}{\big[\big(\theta(k+\gamma)\big)\big]^{m+j+2}}+\frac{\boldsymbol{\Gamma}\big(m+j+3,\big(\theta(k+\gamma)\big)t\big)}{\big(\theta(k+\gamma)\big)^{m+j+3}}\right]$$

$$+\delta\left[\frac{\acute{\Gamma}\big(m+j+2,(\theta(k+\gamma))t\big)(\theta(k+\gamma))}{\big[(\theta(k+\gamma))\big]^{m+j+2}}+\frac{\acute{\Gamma}\big(m+j+3,(\theta(k+\gamma))t\big)(\theta(k+\gamma))}{(\theta(k+\gamma))^{m+j+3}}\right].$$

We note that $\acute{m}(t)>0,\ t>0$, thus mean of residual life is increasing (IMRL).

3.3. Reversed Residual Life

The *n*th moments of residual life denoted by $E[(t-x)^n|\,x\le t]$ where n = 1, 2, 3, … is defined by

$$M(t)_n=\tfrac{1}{F(T)}\int_0^t(t-x)^n\,f(x)dx.$$

We substitute in the general form for the residual life for the probability density function of the Kumaraswamy Lindley distribution, the reversed residual life becomes

$$M(t)_n=\frac{1}{F(T)}\int_0^{t\infty}(t-x)^n\ \frac{ab\,\theta^2}{\theta\ +\ 1}\ w_j\,q\,x^{m+j}\,(1+x)\,e^{-\theta x(k+\gamma)}dx.$$

Using the binomial expansion of the expression:

$$(t-x)^n=(t)^n(1-\frac{x}{t})^n=(t)^n\sum_{\omega=0}^{\infty}(-1)^{\omega}\binom{n}{\omega}(\frac{x}{t})^{\omega}$$

$$M(t)_n=\frac{1}{F(T)}\int_0^t(t)^n\sum_{\omega=0}^{\infty}(-1)^{\omega}\binom{n}{\omega}(\frac{x}{t})^h\ \frac{ab\,\theta^2}{\theta\ +\ 1}\ w_j\,q\,x^{m+j}\,(1+x)\,e^{-\theta x(k+\gamma)}dx.$$

By simplifying the last equation, we get

$$M(t)_n=\frac{\frac{ab\,\theta^2}{\theta\ +\ 1}\ w_j\,q(t)^n\sum_{\omega=0}^{\infty}\frac{(-1)^{\omega}\binom{n}{\omega}}{t^{\omega}}}{F(T)}\int_0^t x^{\omega}\ x^{m+j}\,(1+x)\,e^{-\theta x(k+\gamma)}dx.$$

Suppose that $\varphi=\frac{\frac{ab\,\theta^2}{\theta\ +\ 1}w_j\,q(t)^n\sum_{\omega=0}^{\infty}\frac{(-1)^{\omega}\binom{n}{\omega}}{t^{\omega}}}{F(T)}$, the last equation becomes

$$M(t)_n=\varphi\int_0^t\ x^{m+j+\omega}\,(1+x)\,e^{-\theta x(k+\gamma)}dx$$

$$M(t)_n = \varphi\left[\int_0^t x^{m+j+\omega}\, e^{-\theta x(k+\gamma)}dx + x^{m+j+\omega+1}e^{-\theta x(k+\gamma)}dx\right].$$

Let $k_8 = \int_0^t x^{m+j+w}\, e^{-\theta x(k+\gamma)}dx$, $k_9 = \int_0^t x^{m+j+\omega+1}e^{-\theta x(k+\gamma)})\ dx,$ the reversed of residual life becomes

$$M(t) = \varphi[\, k_8 + \, k_9].$$

Now, we calculate the value of k_8

$$k_8 = \int_0^t x^{m+j+w}\, e^{-\theta x(k+\gamma)}dx.$$

By assuming that $y = (\theta(k+\gamma))x$, we get

$$k_8 = \int_0^t \left[\frac{y}{(\theta(k+\gamma))}\right]^{m+j+w} e^{-y}\frac{dy}{(\theta(k+\gamma))}$$

$$k_8 = \frac{1}{(\theta(k+\gamma))^{m+j+w+1}}\int_0^{(\theta(k+\gamma))t}[y]^{m+j+w}\, e^{-y}dy.$$

From the general definition of the gamma function $a-1 = \, m+j+w$ and $b = \theta(k+\gamma))b,$

$$k_8 = \frac{\boldsymbol{\Gamma}\,(m+j+w+1, \theta(k+\gamma))b)}{(\theta(k+\gamma))^{m+j+w+1}}$$

We calculate the value of k_9

$$k_9 = \int_0^t x^{m+j+\omega+1}e^{-\theta x(k+\gamma)})\ dx.$$

By assuming that $y = (\theta(k+\gamma))x$, we have

$$k_9 = \int_0^{(\theta(k+\gamma))t}\left[\frac{y}{(\theta(k+\gamma))}\right]^{m+j+\omega+1} e^{-y})\ \frac{dy}{(\theta(k+\gamma))}$$

$$k_9 = \frac{1}{(\theta(k+\gamma))^{m+j+\omega+2}}\int_0^{(\theta(k+\gamma))t}[y]^{m+j+\omega+1}(e^{-y})dy.$$

From the general definition of the gamma function $a-1 = \, m+j+w+1$ and $b = \theta(k+\gamma))b,$

$$k_9 = \frac{\boldsymbol{\Gamma}\,(m+j+w+2, \theta(k+\gamma))b)}{(\theta(k+\gamma))^{m+j+\omega+2}}.$$

The reversed of residuals life becomes

$$M(t) = \varphi\left[\frac{\boldsymbol{\Gamma}\,(m+j+w+1, \theta(k+\gamma))b)}{(\theta(k+\gamma))^{m+j+w+1}} + \frac{\boldsymbol{\Gamma}\,(m+j+w+2, \theta(k+\gamma))b)}{(\theta(k+\gamma))^{m+j+\omega+2}}\right].$$

The cumulative hazard function (CHF) of the Kumaraswamy Lindley distribution denoted by $CHF_{KWL}\,(x,\theta)$ is

$$CHF_{KWL}\,(x,\theta) = \int_0^x h(x,\theta)dx.$$

By substituting for hazard rate function, we get

$$CHF_{KWL}\,(x,\theta) = \int_0^x \frac{f(x)}{1-F(x)}dx.$$

Substituting the value of the probability density function for the Kumaraswamy Lindley distribution as well as the cumulative distribution function for it, we have

$$CHF_{KWL}\,(x,\theta) = \int_0^x \frac{\frac{ab\,\theta^2}{\theta+1} w_j\, q\, x^{m+j}\,(1+x)\, e^{-\theta x(k+\gamma)}}{\left[1-\left(1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+1}\right)\right)^a\right]^b}dx.$$

Applying the rule of integration which states that if the numerator is the differentiation of the denominator, then the integral is Lin (the denominator)

$$CHF_{KWL}\,(x,\theta) = -lin\left[1-\left(1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+1}\right)\right)^a\right]^b.$$

Reverse hazard rate function is

$$\frac{f_{KWL}}{F_{KWL}} = \frac{\frac{ab\,\theta^2}{\theta\;+\;1}\; w_j\; q\; x^{m+j}\;(1+x)\; e^{-\theta x(k+\gamma)}}{1\;-\;\left[1-\left(1-e^{-\theta x}\left(1+\frac{\theta x}{\theta\;+1}\right)\right)^a\right]^b}.$$

Dispersion in a population can be measured by measuring the sum of deviations from the mean and median. If we have X as a random variable that is Kumaraswamy Lindley distributed, we can derive the mean deviation about the mean $\mu = E(x)$ and the mean deviation about the median $M = F^{-1}(\frac{1}{2})$,

$$\delta_1 = \int_0^\infty |X-\mu|\; f_{kwl}(x)dx$$

$$\delta_1 = \int_0^\mu |X-\mu|\; f_{kwl}(x)dx + \int_\mu^\infty (x-\mu)f_{kwl}\,(x)dx$$

$$\delta_1 = \mu\int_0^\mu f_{kwl}(x)dx - \int_0^\mu x\; f_{kwl}(x)dx + \int_\mu^\infty\; xf_{kwl}\,(x)dx - \int_\mu^\infty\; \mu f_{kwl}(x)dx$$

$$\delta_1 = \mu F(\mu) - \int_0^\infty x\; f_{kwl}(x)dx + \int_\mu^\infty\; xf_{kwl}\,(x)dx - \mu\int_\mu^\infty\; f_{kwl}(x)dx.$$

By substituting the value of the integral $\mu\int_\mu^\infty f_{kwl}(x)dx = \mu(1-F(\mu))$, we have

$$\delta_1 = \mu F(\mu) - \int_0^\infty x\; f_{kwl}(x)dx + \int_\mu^\infty\; xf_{kwl}\,(x)dx - \mu(1-F(\mu))$$

$$\delta_1 = 2\mu F(\mu) - 2\mu + \int_\mu^\infty\; xf_{kwl}\,(x)dx.$$

Now, we calculate the value of the integral $\pi = \int_\mu^\infty\; xf_{kwl}\,(x)dx$. By substituting the probability density function for the Kumaraswamy Lindley distribution, we get

$$\pi = \int_\mu^\infty\; x\frac{ab\,\theta^2}{\theta\;+\;1}\; w_j\; q\; x^{m+j}\;(1+x)e^{-\theta x(k+\gamma)}dx(x)dx.$$

Simplifying the previous integration, we have

$$\pi = \frac{ab\,\theta^2}{\theta\;+\;1}\; w_j\; q\int_\mu^\infty\; x^{m+j+1}\;(1+x)\; e^{-\theta x(k+\gamma)}dx(x)dx$$

$$\pi = \frac{ab\,\theta^2}{\theta\;+\;1}\; w_j\; q\left[\int_\mu^\infty\; x^{m+j+1}e^{-\theta x(k+\gamma)}dx + \int_\mu^\infty x^{m+j+2}\; e^{-\theta x(k+\gamma)}dx\right].$$

Suppose that $k_{10} = \int_\mu^\infty\; x^{m+j+1}e^{-\theta x(k+\gamma)}dx$ and $k_{11} = \int_\mu^\infty x^{m+j+2}\; e^{-\theta x(k+\gamma)}dx$, then

$$\pi = \frac{ab\,\theta^2}{\theta\;+\;1}\; w_j\; q\;[\,k_{10} + k_{11}].$$

We calculate the value of k_{10}

$$k_{10} = \int_\mu^\infty\; x^{m+j+1}e^{-\theta x(k+\gamma)}dx$$

$$k_{10} = \int_{\mu}^{\infty} \left[\frac{y}{(\theta(k+\gamma))}\right]^{m+j+1} e^{-y} \frac{dy}{(\theta(k+\gamma))}$$

$$k_{10} = \frac{1}{(\theta(k+\gamma))^{m+j+2}} \int_{\mu(\theta(k+\gamma))}^{\infty} [y]^{m+j+1} e^{-y} dy.$$

Applying the general form of the gamma function, we have

$$k_{10} = \frac{\boldsymbol{\Gamma}(m+j+1, \mu(\theta(k+\gamma)))}{(\theta(k+\gamma))^{m+j+2}}.$$

The value of k_{11} is

$$k_{11} = \int_{\mu}^{\infty} x^{m+j+2}\, e^{-\theta x(k+\gamma)} dx$$

$$k_{11} = \int_{\mu(\theta(k+\gamma))}^{\infty} \left[\frac{y}{(\theta(k+\gamma))}\right]^{m+j+2} e^{-y} \frac{dy}{(\theta(k+\gamma))}$$

$$k_{11} = \frac{1}{[\theta(k+\gamma)]^{m+j+3}} \int_{\mu(\theta(k+\gamma))}^{\infty} [y]^{m+j+2}\, e^{-y}\, dy.$$

Applying the general form of the gamma function, we get

$$k_{11} = \frac{\boldsymbol{\Gamma}(m+j+3, \mu(\theta(k+\gamma)))}{[\theta(k+\gamma)]^{m+j+3}}$$

$$\pi = \frac{ab\,\theta^2}{\theta + 1}\, w_j\, q\, [\, k_{10} + k_{11}].$$

Substituting for the value of $k_{11,}\, k_{10}$ we have

$$\pi = \frac{ab\,\theta^2}{\theta + 1}\, w_j\, q \left[\frac{\boldsymbol{\Gamma}(m+j+1, \mu(\theta(k+\gamma)))}{(\theta(k+\gamma))^{m+j+2}} + \frac{\boldsymbol{\Gamma}(m+j+3, \mu(\theta(k+\gamma)))}{[\theta(k+\gamma)]^{m+j+3}} \right].$$

Substituting for the value π we get the mean division

$$\delta_1 = 2\mu F(\mu) - 2\mu$$

$$+ \frac{ab\,\theta^2}{\theta + 1}\, w_j\, q \left[\frac{\boldsymbol{\Gamma}\left(m+j+1, \mu(\theta(k+\gamma))\right)}{\left(\theta(k+\gamma)\right)^{m+j+2}} + \frac{\boldsymbol{\Gamma}\left(m+j+3, \mu(\theta(k+\gamma))\right)}{[\theta(k+\gamma)]^{m+j+3}} \right].$$

4. CONCLUSIONS

In this research paper, we present some of new mathematical properties of the Kumaraswamy Lindley distribution. The properties of reversed residual life, mean of residual life, moments of residual life, probability weighted moments, and cumulative hazard rate function, have been derived. Furthermore, we invite researchers to study more mathematical properties of the distribution because of its many applications which can contribute to solving many life problems.

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