



THE MODIFIED GLAS-MOSEL FORMULA FOR NUMERICAL INVESTIGATION OF THE FUSION CROSS-SECTIONS OF $16O + 70,72,73,74,76Ge$ - A PRELIMINARY STUDY

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ABSTRACT

Many intense experimental and theoretical studies have been performed to understand the behavior of fusion reactions, especially related to the barrier height of the interacting nuclei. This preliminary study would discuss a variation of the applicability of the Glas-Mosel formula with a little bit of modification applied to heavy systems. The modified Glas-Mosel formula has been utilized to calculate the fusion cross-sections of $16O + 70,72,73,74,76Ge$. To perform the differential and the optimization numerics, the Finite Difference and Nelder-Mead methods were applied to Fortran script-code respectively to assist the computational process. Referring to the obtained results, it can be indicated that the obtained results are in positive agreement with the experimental data. In addition, the modified Glas-Mosel formula proposed in this study has the capability to explain the experimental results or in predicting the fusion cross-section of nuclei. Further investigations are needed to get the crucial data to serve as a basic reference.

Keywords: Glas-Mosel; fusion; cross-section; germanium

INTRODUCTION

Many experiments and theoretical research have been performed to determine the barrier height between the interacting nuclei. Unfortunately, the lack of experimental data or the challenge of measuring cross-sections inside an experiment is why the data are not accessible. Therefore, the formula is needed to solve these problems in providing the data. Several studies have been performed to investigate the fusion cross-section by using experimental^{[1]-[6]} and theoretical^{[7]-[10]}. Some methods have been utilized to explain the experimental results, i.e. Classical Molecular Dynamics^[11], Time Dependent Hartree Fock^[12], Relativistic Mean Field^{[13], [14]}, and Density Constrain Time Dependent Hartree Fock^[15]. Similar to the aim above, in 1974, D. Glas and U. Mosel^[16] launched an expression that explained the constraint on the complete fusion of heavy-ion reaction. This expression can analyze the fusion cross-section data, for both high and low energies, for heavy-ion collisions. It has been effectively utilized to explain experimental research^{[17], [18]}. Using the modified Glas-Mosel formula and referring to the previous study^[19], this study has continued to research the fusion cross-section.

Based on the obtained results of this previous study^[19], it is intended to apply that formula to the heavy nuclei that have experimental data for comparison. In this work, it has been performed a preliminary study to investigate the applicability of this modified formula in

calculating the fusion cross-section of $16O + 70,72,73,74,76Ge$ systems at $36 \leq E \leq 50$ MeV of energies. The germanium fusion has been studied by these researchers^{[20], [21]}. The range energy is based on the experimental data provided by Aguilera *et al.*^[17] It should be noticed that the goal of this study not to examine systematically the Glas-Mosel formula itself but to discuss a variation of the applicability of the Glas-Mosel formula with little bit modification applied to heavy systems and to compare the obtained results with the experimental data provided by the reference^[17]. It is intended to get a semi-empirical formula that can be utilized to explain the experimental data or to predict the statistical data of fusion cross-section.

In this paper, the theory and numerical method are presented briefly in the next section, followed by the result and discussion. Finally, the summary of this paper was explained in the last section.

THEORY AND NUMERICAL METHOD

The pioneering study of D. Glas and U. Mosel led to the expression^[16] which expressed the fusion cross-section as a function of the energy E , i.e.

$$\sigma_F(E) = \frac{\hbar\omega R_B^2}{2E} \ln \ln \left\{ \frac{1 + \exp \exp [2\pi(E - V_B)/\hbar\omega]}{1 + \exp \exp \{2\pi[E - V_B - (E - V_C)R_C^2/R_B^2]/\hbar\omega\}} \right\} \quad (1)$$

where R_B (R_C) represents the barrier (critical) distance, E represents the energy, and V_C represents the critical potential. Here, the subscript F , B , and c refer to the fusion, the barrier, and the critical, respectively. The peak of the total potential is utilized to determine V_B . The parameter of $\hbar\omega$, a representation of the behavior defining the fusion cross-section at extremely low energy close to and below the Coulomb barrier, can be determined by using^[22]

$$\hbar\omega_B = \hbar \left[\frac{1}{\mu} \frac{d^2 V(r)}{dr^2} \Big|_{R_B} \right]^{1/2} \quad (2)$$

where μ is the reduced mass of nuclei. The parameter of $\hbar\omega$ (l -independent) explains the curvature of the parabolic barrier. In this study, the interacting nuclei were treated as the spherical nuclei at the ground-state level. The potential of the interacting nuclei $V(r)$ can be calculated by using^{[23], [24]}

$$V(r) = \frac{Z_p Z_T e^2}{r} + \frac{-V_0}{\left[1 + \exp \left(\frac{r - R_0}{a} \right) \right]} \quad (3)$$

In Eq. (3), Coulomb potential comes in first and the Woods-Saxon potential comes in second. Here Z is the atomic charge, e is the electron charge, V_0 is the potential depth, r is the distance between the interacting nuclei, R_0 is the radius parameter, and a is the surface diffuseness parameter. The subscript p and T refer to respectively the projectile and the target. The potential depth V_0 is described by^{[16], [23], [25], [26]}

$$V_0 = 16\pi\gamma R a \quad (4)$$

and^{[16], [23], [25], [26]}

$$\gamma = \gamma_0 \left[1 - k \left(\frac{(N_T - Z_T)(N_p - Z_p)}{A_T A_p} \right) \right] \quad (5)$$

$$\underline{R} = \frac{R_T R_p}{R_T + R_p} \tag{6}$$

$$R_{p(T)} = 1.23A_{p(T)}^{1/3} - 0.98A_{p(T)}^{-1/3} \tag{7}$$

$$R_0 = r_0(A_p^{1/3} + A_T^{1/3}) \tag{8}$$

$$R_c = r_c(A_p^{1/3} + A_T^{1/3}) \tag{9}$$

$$V_c = \frac{Z_p Z_T e^2}{R_c} \tag{10}$$

In this study, the Eq. (1) has been modified to be

$$\sigma_F(E) = \frac{\hbar\omega_B R_B^2}{2E} \ln \ln \left\{ \frac{1 + \exp \exp [2\pi(E - V_B)/\hbar\omega_B]}{1 + \exp \exp \{2\pi[C_1(E - V_c)R_c^2/R_B^2]/\hbar\omega_B\}} \right\} \tag{11}$$

where C_1 is the parameter. The parameters of V_B and R_B are determined by finding the peak of total potential and its distance respectively. All calculations (Eq. (2)–Eq. (11)) were performed numerically with Fortran script-code. To solve the numerical differential, the Finite Difference Method was adopted as shown in the following formula[27]

$$\frac{d^2y}{dx^2} = \frac{y(x + \Delta x) - 2y + y(x - \Delta x)}{(\Delta x)^2} \tag{12}$$

where Δx represents the step of the axis. To assist in optimizing the parameters of a , r_0 , r_c , and C_1 , it is utilized the Nelder-Mead method.

Table 1. The algorithm of Nelder-Mead method^{[28], [32]}.

Case (i)	
Determine vertex as B , G , and W	
Compute	
$M = \frac{(B+G)}{2}$, $R = 2M - W$, and $E = 2R - M$	
If $f(R) < f(G)$, then	
Perform case (ii) → either reflect or extend	
Else	
Perform case (iii) → either contract or shrink	
Case (ii)	Case (iii)
Begin	Begin
If $f(B) < f(R)$ then	If $f(B) < f(W)$ then
Replace W with R	Replace W with R
Else	Compute $C = \frac{(W+M)}{2}$
Compute E and $f(E)$	or $C = \frac{(R+M)}{2}$ and $f(C)$
If $f(E) < f(B)$ then	If $f(C) < f(W)$ then
Replace W with E	Replace W with C
Else	Else
Replace W with R	Compute S and $f(S)$
End if	Replace W with S
End if	Replace G with M
End Case (ii)	End if
	End Case (iii)

This method can improve computational problem optimization or solving the analytic problems with an unknown gradient^[28], which has been proven successfully in optimizing two^[29] or more^{[19], [30], [31]} variational parameters. The algorithm of Nelder-Mead method can be seen in Table 1 and the great explanations can be found in these references^{[28], [32]}. The chi-square can be calculated to ensure the precision of calculation by using^{[21], [33], [34]}

$$\chi^2 = \frac{1}{N} \sum_{i=1}^N \left[\frac{(\sigma_i^{theory} - \sigma_i^{exp})}{\Delta\sigma_i^{exp}} \right]^2 \quad (13)$$

where the measurement deviation is represented by $\Delta\sigma_i^{exp}$ and the amount of data is represented by N . The quality of the fit can be ensured by the minimum value of χ^2 . The smaller the value of χ^2 , the higher the quality of the fit^[34].

RESULTS AND DISCUSSION

This study commenced the calculation by optimizing the parameters of a , r_0 , r_c , and C_1 in the Nelder-Mead framework with Fortran style-code to minimize the chi-square in obtaining the fusion cross-section value. To ensure calculation saturation, the chi-square tolerance for each system was set 10^{-20} . The calculation results of this study can be found in Table 2. It can be seen that the obtained chi-squares are less than five for each system, which indicates a successful achievement of a good agreement between this computation and the experiment results. The model of the used potential for each system can be found in Figure 1. Each system investigated in this study has a similar model to each other.

Furthermore, it has been extracted the experimental data from the paper of Aguilera *et al.*^[17] and compare it with our obtained results. The experimental data has been utilized to guarantee that the calculation result of this study have proceeded in the correct procedure. The obtained results of this study and the experimental results of Aguilera *et al.* have been appeared in Figure 2–**Figure 6**. These figures show the fusion cross section of each system as an energy function displayed on both linear and logarithmic. The acquired results of this investigation are in strong accord with the experiment results of Aguilera *et al.*, as can be seen from these figures.

Table 2. Calculation results.

Parameters	Systems				
	160 + 70Ge	160 + 72Ge	160 + 73Ge	160 + 74Ge	160 + 76Ge
a (fm)	0.6383	0.6385	0.6388	0.6391	0.6392
r_0 (fm)	1.1901	1.1903	1.1904	1.1906	1.1910
r_c (fm)	1.5743	1.5745	1.5746	1.5748	1.5751
C_1	0.6375	0.6382	0.6413	0.6451	0.6475
R_0 (fm)	7.9036	7.9512	7.9747	7.9987	8.0460
R_B (fm)	9.8	9.9	9.9	9.9	10.0
R_c (fm)	10.4551	10.5177	10.5485	10.5798	10.6408
V_0 (MeV)	52.9248	53.1371	53.2577	53.3768	53.5690
V_B (MeV)	35.0352	34.8376	34.7531	34.6431	34.4569
V_c (MeV)	35.2584	35.0487	34.9557	34.8429	34.6430
$\hbar\omega_B$ (MeV)	3.8528	3.7154	3.7821	3.8493	3.7114
χ^2	0.9697	4.6797	1.7147	0.9387	2.3316

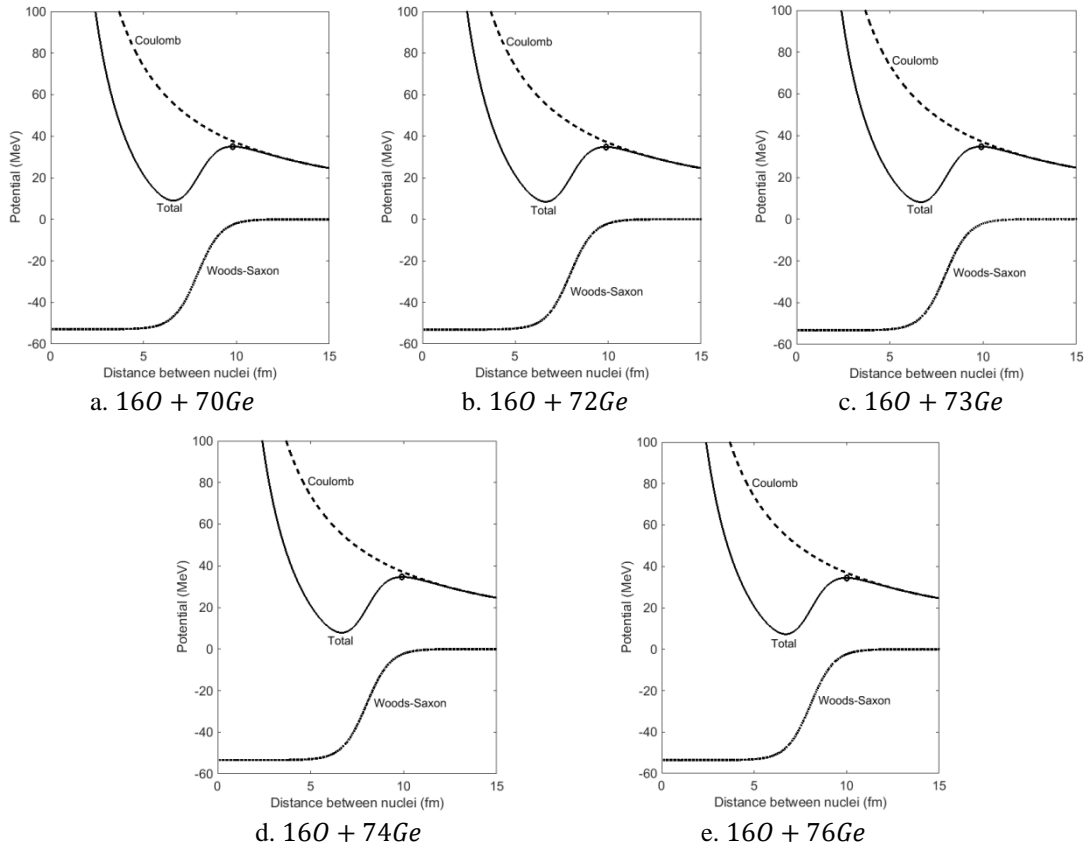


Figure 1. The potential model for each system.

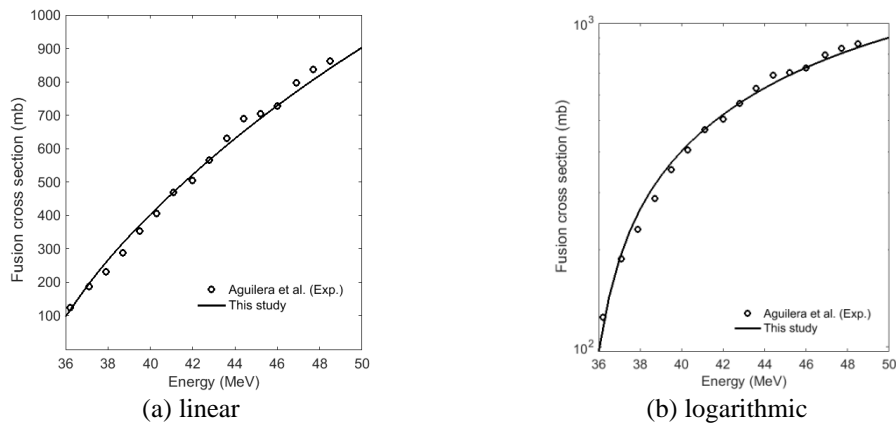


Figure 2. The fusion cross-sections for $16O + 70Ge$, obtained by Aguilera *et al.* [17] and this study.

The aforementioned results show that the modified Glas-Mosel formula performs a good competence in explaining the experiment results of the fusion cross-sections of $16O + 70,72,73,74,76Ge$. It infers that the modified Glas-Mosel formula can be a piece of beneficial equipment in explaining the experimental results. In addition, this expression can provide the prediction of nuclear data by calculation.

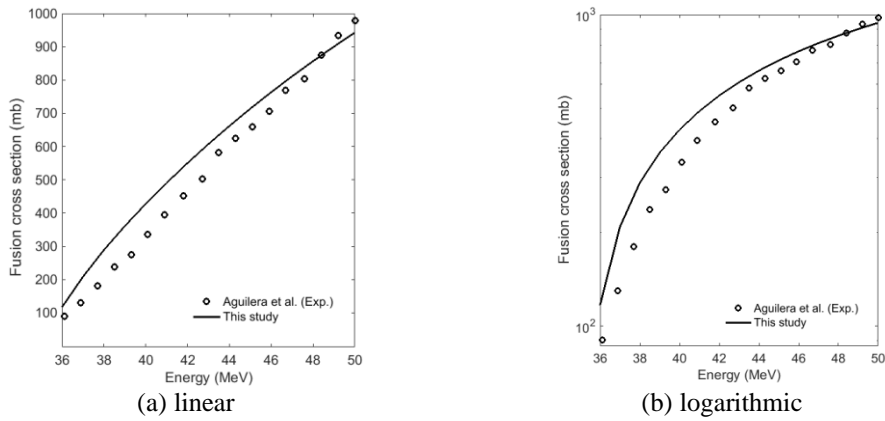


Figure 3. The fusion cross-sections for $160 + 72Ge$, obtained by Aguilera *et al.* [17] and this study.

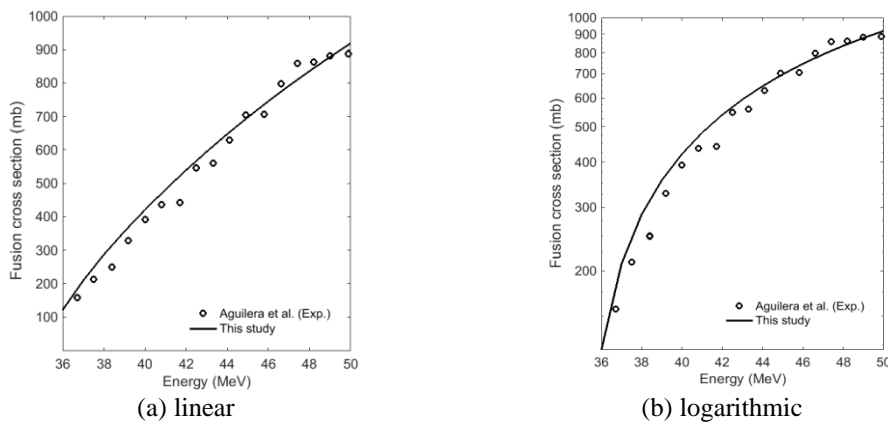


Figure 4. The fusion cross-sections for $160 + 73Ge$, obtained by Aguilera *et al.* [17] and this study.

Further investigations are necessary to be performed for heavy systems, especially at low and high energies. In order to perform further research on the fusion cross-section using heavy or super-heavy nuclei, particularly for theoretical prediction, additional investigations are intended to provide significant information as a fundamental reference. In addition, the obtained results of this study provide the cross-section data in a standard way in which they can be compared to each other.

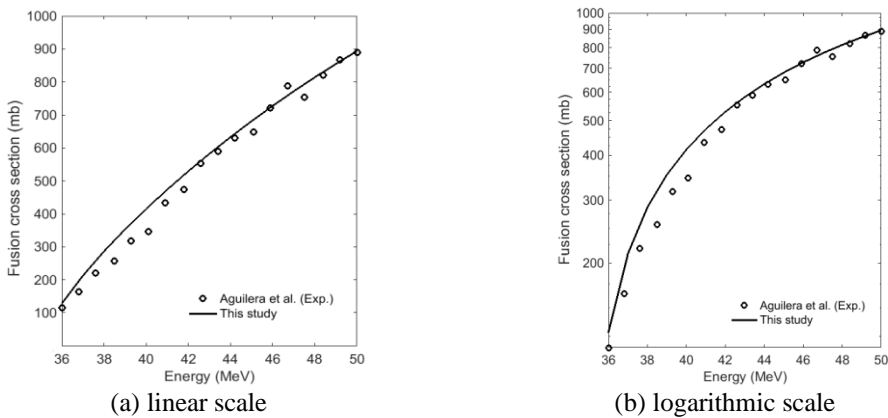


Figure 5. The fusion cross-sections for $160 + 74Ge$, obtained by Aguilera *et al.* [17] and this study.

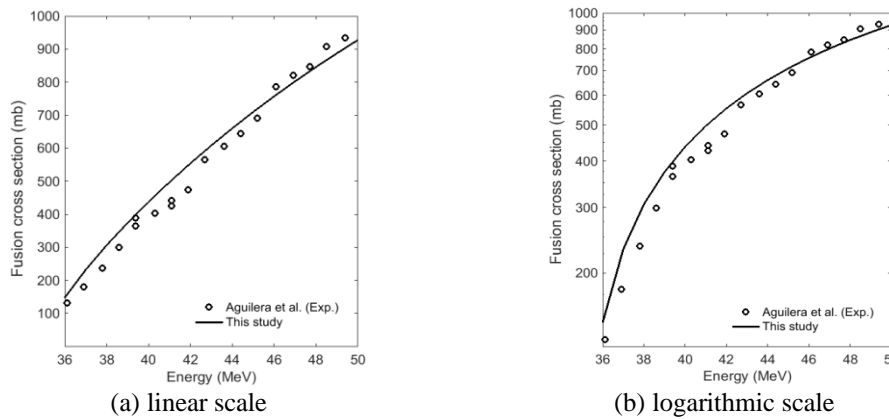


Figure 6. The fusion cross-sections for $160 + 76\text{Ge}$, obtained by Aguilera *et al.* [17] and this study.

CONCLUSION

It has been carried out the study of numerical analysis of fusion cross-section for $160 + 70, 72, 73, 74, 76\text{Ge}$ by using the modified Glas-Mosel formula. The energies are specified at $36 \leq E \leq 50$ MeV. The potentials of the interacting nuclei were approached by using the Woods-Saxon and the Coulomb potentials. The methods of finite difference and Nelder-Mead were utilized to assist the calculation performed by using Fortran script-code. At these selected energies, the obtained results achieve a good agreement with the experimental data. Referring to these results above, it can be indicated that the modified Glas-Mosel formula used in this study is a potential candidate to be a valuable tool for explaining the experimental results or for predicting the fusion cross-section of nuclei. It needs more research to get the crucial data needed to serve as a basic reference.

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