Weber's electrodynamics for the hydrogen atom

H. Torres-Silva [1], J. López-Bonilla [2], R. López-Vázquez [2], J. Rivera-Rebolledo [3]

[1] Escuela de Ing. Eléctrica y Electrónica, Univ. de Tarapacá, Arica Casilla 6-D, Chile
[2] ESIME-Zacatenco, IPN, Col. Lindavista CP 07738, México DF; jlopezb@ipn.mx
[3] Depto. Física, ESFM-IPN, Edif. 9, Col. Lindavista CP 07738, México DF

ABSTRACT

The original Weber action at a distance theory is valid for slowly varying effects, and it in addition to predicting all of the usual electrodynamical results, leads to crucial effects where the Maxwell theory fails. The Weber's approach is an alternative to Maxwell electrodynamics, where the Coulomb's law becomes velocity dependent [1-6]. Here we prove that the Weber's theory gives the fine structure energy level splitting for the hydrogen atom without the assumption of mass change with velocity.

2010 Mathematical Subject Classification: 35Q61, 78A25, 81V10
Keywords: Weber electrodynamics, Hydrogen atom, Energy level splitting, Lamb shift

INTRODUCTION

In Maxwell electrodynamics, Newton's third law does not hold for particles. Instead, particles exert forces on electromagnetic fields, and fields exert forces on particles, but particles do not *directly* exert forces on other particles. Therefore, two nearby particles need not experience equal and opposite forces. Related to this, Maxwell electrodynamics predicts that the laws of conservation of momentum and conservation of angular momentum are valid *only* if the momentum of particles *and* the momentum of surrounding electromagnetic fields are taken into account. The total momentum of all particles is not necessarily conserved, because the particles may transfer some of their momentum to electromagnetic fields or vice-versa. The well-known phenomenon of radiation pressure proves that electromagnetic waves are indeed able to "push" on matter, see Maxwell stress tensor and Poynting [7] vector for further details. The Weber force law is quite different: All particles, regardless of size and mass, will exactly follow Newton's third law. Therefore, Weber electrodynamics, unlike Maxwell theory, has conservation of *particle* momentum and conservation of *particle* angular momentum.

The main point to be noted about the Lagrangian method is that when potential energy depends only on position of the particle described (here we shall simplify by considering a one-particle system) there is a direct relationship between the Lagrangian function L and physical quantities, namely, $L = T - V$, where T is the kinetic energy of the particle and V is its physical potential energy. But when potential energy depends on particle velocity as well as position the formalism loses direct contact with physics and requires a Lagrangian function employing a so-called "generalized potential energy" having nothing to do with, and quite distinct from, the

physical potential energy, so V no longer enters L. Then, the physical potential energy must be replaced by some non-physical function which we shall designate as S (following Assis-Caluzi [2]). Thus $L = T - S$ where $S = S(q_k, \dot{q}_k)$, the q's are generalized coordinates and the $\dot{q}$'s are the corresponding generalized velocities. This S-function (not to be confused with Hamilton's principal function [8]) must be contrived to "give the right answer" – meaning that the right answer (the forces or equations of motion) must be known from other considerations. The Hamiltonian H does not suffer of this detachment from physics because it is always equal to the total physical energy, $H = T + V$, and is related to L by a Legendre transform [8-11]:

$$H = \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L .\tag{1}$$

Eq. (1) is often spoken of as the "definition" of H, but that is a mathematician's way of looking at it. The physics is all in H, which always depends on the physical potential energy V; whereas L depends on the non-physical "generalized potential energy" S in the case of a velocity-dependent physical potential energy, $V = V(q_k, \dot{q}_k)$.

Weber's law, in this context, postulates the physical potential energy to be:

$$V_{Weber} = \frac{Ze^2}{4\pi\varepsilon_0} \frac{1}{r} \left(1 - \frac{\dot{r}^2}{c^2} \right)\tag{2}$$

and the "generalized Lagrangian potential" to be $S_{Weber} = \dfrac{Ze^2}{4\pi\varepsilon_0} \dfrac{1}{r} \left(1 + \dfrac{\dot{r}^2}{c^2} \right)$; the Hamiltonian is

with the reduced mass $\mu = m_e$:

$$H_{weber} = \frac{Ze^2}{4\pi\varepsilon_0} \left(\frac{1}{r} - \frac{1}{2m_e c^2} \frac{(p \cdot r)^2}{r^3} \right) .\tag{3}$$

As the Weber theory has no radiation one can think that it can be useful to calculate the energy levels for the hydrogen atom [12, 13] including corrections through order α^2, where α is the fine structure constant [14-18], and compare with the typical corrections through order α^2 obtained from the relativistic Dirac's equation or the Schrödinger's equation with relativistic mass correction and spin-orbit effect. In this paper we show that the Weber electrodynamics automatically gives the fine structure energy level splitting of the hydrogen atom [19-24] without having to make the additional assumption of mass change with velocity

Hydrogen fine structure with Weber theory

Without taking into account the spin of the electron we define:

$$\Delta H_{weber} = -\frac{e^2}{4\pi\varepsilon_0} \frac{1}{2m_e c^2} \frac{(p \cdot r)^2}{r^3} \ , \quad E^0_{n\,weber} = -\frac{e^2}{4\pi\varepsilon_0} \frac{1}{2a_B} \frac{1}{n^2} \ , \quad a_B = \frac{\hbar^2}{m_e \dfrac{Ze^2}{4\pi\varepsilon_0}} \ ,\tag{4}$$

where E^0_n is the unperturbed energy, then:

$$\frac{(p \cdot r)^2}{r^3} = \left(\frac{\nabla^2}{r} - \frac{1}{r^2} \frac{\partial}{\partial r} - \frac{1}{r^3} \Lambda \right) \hbar^2 , \Lambda = -l(l+1) ,$$

$$\Delta H_{weber} = -\frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{m_e c^2}\left(\frac{E_n}{r} + \frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{r^2} + \frac{\hbar^2}{2m_e r^2}\frac{\partial}{\partial r} - \frac{(\hbar)^2 l(l+1)}{2m_e r^3}\right). \tag{5}$$

For the unperturbed state ψ^0 of the Schrödinger equation, we have $\hat{P}^2\psi^0 = 2m_e(E-V)\psi^0$.

Eq. (5) gives a correction for the relativistic velocity of the electron which is of the order of v^2/c^2, then using the right-hand-side of this equation, the energy correction is:

$$\Delta E^1_{weber} = -\int \psi^{0*}\Delta H_{weber}\psi^0 d^3 r, \tag{6}$$

and $<r^{-1}>$ can be calculated with the help of the virial theorem [12, 25, 26]:

$$\langle T\rangle = +\frac{1}{2}\langle \vec{r}\cdot\vec{\nabla}V\rangle, \quad V = \frac{e^2}{4\pi\varepsilon_0 r} \Rightarrow \langle T\rangle = +\frac{1}{2}\left\langle \vec{r}\cdot\left(\frac{-e^2}{4\pi\varepsilon_0}\right)\frac{\hat{r}}{r^2}\right\rangle = -\frac{1}{2}\left\langle \frac{e^2}{4\pi\varepsilon_0 r}\right\rangle.$$

Now $\langle E\rangle = \langle T\rangle + \langle V\rangle = -\frac{1}{2}\langle V\rangle + \langle V\rangle = \frac{1}{2}\langle V\rangle$, but $\langle E\rangle = -\frac{1}{2a_B}\left(\frac{e^2}{4\pi\varepsilon_0}\right)\frac{1}{n^2} \equiv E^0_n$ and

$\langle V\rangle = \frac{-e^2}{4\pi\varepsilon_0}\langle r^{-1}\rangle$, therefore $<r^{-1}>=\frac{Z}{a_B n^2}$. The expectation value $\langle r^{-2}\rangle$ is more difficult,

the result is [25, 27] $<r^{-2}>=\frac{Z^2}{a_B n^3 l(l+2)}$ because $\int_o^\infty Rr^{-2}\frac{dR}{dr}r^2 dr = -1/2|R_0|^2$

and for s states $|R_0|^2 = \frac{4}{a_B^3 n^3}$.

It is traditional to write these energy corrections in terms of the Fine Structure Constant [14-18] $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137.06}$, $E^0_n = -\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 = -\frac{1}{2}m_e c^2\frac{\alpha^2}{n^2}$, $a_B = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} = \frac{\hbar}{m_e\alpha c}$.

For l states $<\Delta H_{weber}>=-\frac{Z^4\alpha^4}{n^3}m_e c^2\left(-\frac{1}{n}+\frac{1}{2(l+1/2)}\right)$. For s states we consider the term

$-\frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{m_e c^2}\left(\frac{\hbar^2}{2m_e r^2}\frac{\partial}{\partial r}\right)$, then $<-\frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{m_e c^2}\left(\frac{\hbar^2}{2m_e r^2}\frac{\partial}{\partial r}\right)>=\frac{Z^4\alpha^4}{n^3}m_e c^2$ so:

$$<\Delta H_{weber}>\big|_{l=0} = -\frac{Z^4\alpha^4}{n^3}m_e c^2\left(-1-\frac{1}{n}+\frac{1}{2(0+1/2)}\right) = \frac{Z^4\alpha^4}{n^3}m_e c^2\left(\frac{1}{n}\right). \tag{7}$$

The equivalent Weber expression for energy level is:

$$E_{nlweber} = -\frac{13.6eV}{n^2}\left[1+\frac{\alpha^2}{n^2}\left(\frac{n}{l+1/2}-2\right)\right], \tag{8}$$

which can be compared with the Klein-Gordon expression or the Schrödinger equation plus the term $-p^4/8m^3c^2$ which gives a correction for the relativistic velocity of the electron of order v^2/c^2:

$$E_{nlklein} = -\frac{13.6eV}{n^2}\left[1+\frac{\alpha^2}{n^2}\left(\frac{n}{l+1/2}-3/4\right)\right].$$ (9)

It can be seen from here that the splitting of levels is proportional to the square of the fine structure constant. Thus:

$$\Delta H_{rweber}^1 = \frac{-\frac{1}{2}m_ec^2\alpha^2}{2m_ec^2n^2}\left|E_n^0\right|\left(\frac{n}{\ell+1/2}-2\right) = -\frac{\alpha^2}{n^2}\left|E_n^0\right|\left(\frac{n}{\ell+1/2}-2\right),$$ (10)

down by a factor of α^2 from the Bohr energies which are themselves proportional to α^2,

of the same order as the relativistic correction of formula (9) which exactly coincides with the expressions for the additional relativistic energy that can be derived in the same approximation with the help of the Klein-Gordon equation.

With the help of the Weber wave equation results the energy spectrum:

$$E_{nl} = -\frac{13.6eV}{n^2}\left[1+\frac{Z^2\alpha^2}{n^2}\left(\frac{n}{l+1/2}-2\right)\right],$$ (11)

where the first term is identical to the corresponding expression in the non-relativistic theory while the second term, which is proportional to the square of the fine structure constant $\alpha = 1/137$, gives the relativistic correction. With this expression (11) we can evaluate the levels $(2p_{3/2})$ $(2p_{1/2})$ $(2s_{1/2})$:

$$\Delta E_{weber}(2p_{3/2}) = \alpha^2E_n(-2/4+(1/2)(1/2)) = \alpha^2E_n(-1/4) = -\alpha^2E_2/4,$$

$$\Delta E_{weber}(2p_{1/2}) = \alpha^2E_n(-1/2+(1/2)(1)) = \alpha^2E_2(0) = 0,$$ (12)

$$\Delta E_{weber}(2s_{1/2}) = \alpha^2E_n(-1/2+(1/2)(1)) = \alpha^2E_2(0) = 0.$$

A consideration of relativistic corrections for the hydrogen atom, $Z=1$, is interesting since it removes degeneracy with respect to l and hence levels with a given value of n spilt up into n closely spaced sublevels, because the orbital quantum l may assume n values ($l = 1, 2, 3.....n-1$).

For the shake of comparison with the experimental result, let us calculate the double splitting for the Balmer series ($n=2$) [28]. With the help of E_{nl} the splitting can be written as

$$(E_{21} - E_{20})/\hbar = \frac{8}{3}\frac{R\alpha^2}{16},$$ and a comparison of results with the experimental data shows that

the real magnitude of the splitting for the Balmer series is about three times less than the value obtained experimentally. The reason behind this discrepancy is that the fine structure of the atomic levels is not exhausted by the term ΔH_{weber} or relativistic dependence of mass on

velocity, so we should also take into account the spin of electron. Obviously the Weber equation is applicable to pi-mesons which are particles without spin, in particular, it can also describe the motion of negative pi-mesons around the nucleus; such pi-mesons have already been observed experimentally.

At the same time, if we use the Klein-Gordon equation the corresponding splitting is equal to

$$\Delta^{(2)}\omega_{KG} = (2s) - (2p) = \frac{8}{3}\frac{R\alpha^2}{16},$$ which is the same result obtained if we take into account

only the relativistic effects (Klein-Gordon equation), see eq. (9). In this connection, it is interesting to note that the fine structure of the hydrogen atom spectrum was first calculated by Sommerfeld [14] from Bohr semiclassical theory by taking the relativistic expression for the Hamiltonian as the starting point. Sommerfeld obtained without taking spin into account

$$\Delta^{(2)}\omega_S = (2s) - (2p) = \frac{R}{16}\alpha^2.$$ He was not able to show three levels for $n=2$ whose existence

was later confirmed experimentally [19-22].

Fine structure analysis according to Weber theory plus spin effects

After some algebra including the spin orbit effect we can write the expression for the energy levels of hydrogen atom including corrections through order α^2. If we take into account the fine structure, the energy levels of a hydrogen atom are found to depend on the internal quantum number j as well. The corresponding terms are equal to:

$$E_{njrel} = -\frac{13.6eV}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - 2\right)\right], \qquad (13)$$

the factor of 2 at the end comes from the electron spin $S=1/2$, see [12]. This modification of the energy levels of a hydrogen atom due to a combination of relativity and spin-orbit coupling is known as *fine structure*. It is clear that from this formula, in according with the Dirac's theory, the fine structure depends only on the principal quantum number n and the inner quantum number j. Unlike the Klein-Gordon theory which does not take into account, the fine structure of levels does not depend on the orbital quantum number l in this case. It can be seen in figure 1 that all levels are split into two sublevels, since two values of j correspond to each value of l. For example, instead at the simple level $2p$ ($l=1$), we now have two levels $^2P_{1/2}$ and $^2P_{3/2}$. The only exception is the s-level ($l=0$) for which j can assumes only one value ($j=1/2$).

It should be noted that due to splitting of the energy levels, the degeneracy undergoes a change. Instead, the principal quantum number n assume the following values: $n=1, 2, 3$. The orbital quantum number l can change from $l=0$ to $l=n-1$. The inner quantum number j assumes the values $j=\pm 1/2$ ($l \neq 0$) and $j=1/2$ ($l=0$). The fine structure breaks the degeneracy in ℓ, and the energies are determined by n and j. Fig. 1 shows this effect. Coincidentally, the corrections add up in such a way that the $^2P_{1/2}$ and $^2S_{1/2}$ states remain degenerate.
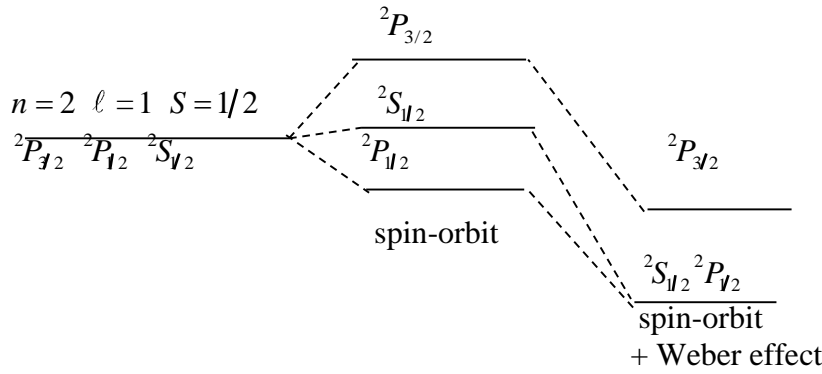
Figure 1. Splitting of energy levels of the hydrogen atom using the Weber theory plus spin orbit effects.

Actually the $^2S_{1/2}$ and $^2P_{1/2}$ levels are degenerate even under the complete relativistic Dirac equation treatment and the Weber approach. While determining the magnitude of splitting of spectral lines, we must take into account the selection rules. Then, instead of one line in the Lyman series [29], we shall now get two lines $\Delta^{(1)}\omega = (1s_{1/2}) - (np_{1/2})$ a weak intensity line since $\Delta j = 0$, and $\Delta^{(2)}\omega = (1s_{1/2}) - (np_{3/2})$.

For the Balmer series line we get the following splitting:

$$\Delta^{(1)}\omega = (2s_{1/2}) - (np_{1/2}), \quad \Delta^{(2)}\omega = (2s_{1/2}) - (np_{3/2}), \quad \Delta^{(3)}\omega = (2p_{1/2}) - (ns_{1/2}),$$

$$\Delta^{(4)}\omega = (2p_{1/2}) - (nd_{3/2}), \quad \Delta^{(5)}\omega = (2p_{3/2}) - (ns_{1/2}), \quad \Delta^{(6)}\omega = (2p_{3/2}) - (nd_{3/2}),$$

$$\Delta^{(7)}\omega = (2p_{3/2}) - (nd_{5/2}),$$

where the line $(2p_{3/2}) - (nd_{3/2})$ must be missing since in this case delta $j=2$ (forbidden transition). It should be observed that if the degeneracy with respect to l is not removed, the lines $\Delta^{(1)}\omega$ and $\Delta^{(3)}\omega$ (as well as $\Delta^{(2)}\omega$ and $\Delta^{(4)}\omega$) coincide with each other, since the initial and final levels have the same value for the principal quantum number n and the inner quantum number j; similary, we can determine the law of splitting for other lines. In this case, the lowest energy level which undergoes splitting is the level with $n=2$. For a hydrogen atom (Z=1), the splitting of this level was experimentally investigated quite extensively. The level $n=2$ must split into three levels, and in concordance with the theory described above, two of these levels coincide:

$$(2s_{1/2}) = (2p_{1/2}) = \frac{R}{4}\left[1 + \frac{\alpha^2}{4}(2-2)\right], \quad (2p_{3/2}) = \frac{R}{4}\left[1 + \frac{\alpha^2}{4}(1-2)\right].$$

In accordance with the Dirac's theory we get $\Delta^{(2)}\omega = (2p_{1/2}) - (2p_{3/2}) = \frac{R}{16}\alpha^2$ which is found to be equal to $1.095 x 10^4$ MHz (H fine structure).

Also, for $^{2}S_{1/2}$ and $^{2}P_{1/2}$ the degeneracy is lifted (slightly) by quantum electrodynamics due to the interaction of the electron with its own electromagnetic field. The $^{2}S_{1/2}$ level shifts upward ever so slightly, this is known as the Lamb shift [30-33], and it is of order $m_{e}c^{2}\alpha^{5}\log\alpha$:

$$2^{2}S_{1/2} - 2^{2}P_{1/2} = 1057.862 \ MHZ \quad \text{(microwave range Lamb shift).}$$

REFERENCES

1. A. K. T. Assis, H. Torres-Silva, *Comparison between Weber's electrodynamics and classical electrodynamics*, Pramana J. Phys. **55** (3) (2000) 393–404

2. A. K. T. Assis, J. J. Caluzi, *A limitation of Weber's law*, Phys. Lett. A**160** (1) (1991) 25–30

3. J. P. Wesley, *Weber electrodynamics*, *I: General theory*, *steady current effects*, Found. Phys. Lett. **3** (5) (1990) 443–469

4. J. P. Wesley, *Weber electrodynamics*, *III: Mechanics, gravitation*, Found. Phys. Lett. **3** (5) (1990) 581–605

5. D. K. Lemon, W. F. Edwards, C. S. Kenyon, *Electric potentials associated with steady currents in superconducting coils*, Phys. Lett. A**162** (2) (1992) 105–114

6. D. R. Walz, H. R. Noyes, *Calorimetric test of special relativity*, Phys. Rev. A**29** (1) (1984) 2110-2114

7. J. H. Poynting, *On the transfer of energy in the electromagnetic field,* Phil. Trans. Roy. Soc. London **175** (1884) 343-361

8. C. Lanczos, *The variational principles of mechanics,* University of Toronto Press (1970)

9. A. M. Legendre, *L'intégration de quelques equations aux differences partielles,* Mémoires de l'Académie (1787)

10. L. Tisza, *Generalized thermodynamics,* MIT Press, Cambridge MA (1966)

11. R. K. P. Zia, E. F. Redish, S. R. McKay, *Making sense of the Legendre transform,* Am. J. Phys. **77**, No. 7 (2009) 614-622

12. D. Griffiths, *Introduction to quantum mechanics,* Prentice Hall, New Jersey (2004) Chap. 6

13. J. J. Sakurai, J. J. Napolitano, *Modern quantum mechanics,* Addison-Wesley, New York (2010)

14. A. Sommerfeld, *Zur quantentheorie der spektrallinien,* Ann. der Physik **51** (1916) 1-94

15. M. Born, *The mysterious number 137,* Proc. Indian Acad. Sci. A**2** (1935) 533-561

16. T. Kinoshita, *The fine-structure constant,* Rep. Prog. Phys. **59** (1996) 1459-1492

17. H. Kragh, *Magic number: A partial history of the fine structure constant,* Arch. Hist. Exact Sci. **57**, No. 5 (2003) 395-431

18. V. Tarasenko, *The fine structure constant: its physical nature,* Galilean Electrodynamics **15**, SI, No. 1 (2004) 2

19. A. A. Michelson, E. W. Morley, Phil. Mag. **24** (1887) 46

20. W. Houston, Y. M. Hsieh, *The fine structure of the Balmer lines,* Phys. Rev. **45** (1934) 263-272

21. W. Houston, *A new method of analysis of the structure of H_α and D_α,* Phys. Rev. **51**, No. 6 (1937) 446-449

22. R. C. Williams, *The fine structure of H_α and D_α under varying discharge conditions,* Phys. Rev. **54** (1938) 558-567

23. S. Pasternack, *A note on the fine structure of H_α and D_α,* Phys. Rev. **54** (1938) 1113

24. H. Kragh, *The fine structure of hydrogen and the gross structure of the physics community, 1916-26,* Hist. Stud. Phys. Sci. **15** (1985) 67-126

25. J. López-Bonilla, J. Morales, M. Rosales, *Hypervirial theorem and matrix elements for the Coulomb potential,* Int. J. Quantum Chem. **53**, No. 1 (1995) 3-7

26. J. López-Bonilla, H. N. Núñez-Yépez, A. L. Salas-Brito, *Generalized hypervirial and recurrence relations for hydrogenics matrix elements,* J. Phys. B: At. Mol. Opt. Phys. **28**, No. 17 (1995) L525-L529

27. M. Bautista M., J. López-Bonilla, J. Morales, *Matrix elements for the Coulomb interaction,* Apeiron **13**, No. 1 (2006) 34-42

28. J. J. Balmer, *Notiz über die spektralinien des wasserstoffs,* Verhandlungen der Naturforschenden Gesellschaft in Basel **7** (1885) 548-560

29. T. Lyman, *The spectrum of hydrogen in the region of extremely short wave-length,* Memoirs Am. Acad. Arts Sci. **13**, No. 3 (1906) 125-146

30. W. E. Lamb, R. C. Retherford, *Fine structure of the hydrogen atom by a microwave method,* Phys. Rev. **72**, No. 3 (1947) 241-243

31. H. A. Bethe, *The electromagnetic shift of energy levels,* Phys. Rev. **72**, No. 4 (1947) 339-341

32. N. M. Kroll, W. E. Lamb, *On the self-energy of a bound electron,* Phys. Rev. **75**, No. 3 (1949) 388-398

33. W. E. Lamb, *The fine-structure of hydrogen,* in 'The Birth of Particle Physics', Eds. Laurie M. Brown, Lillian Hoddeson, Cambridge University Press (1983) 311-328