Numerical Analysis of Fusion Cross Section of $^{16}\mathrm{O} + {}^{16}\mathrm{O}$ by Using The Modified Glas-Mosel Formula

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ABSTRACT

One of the interesting topics in nuclear reactions is a study about reaction cross section between the interacting nuclei. For calculating fusion cross section, the Glas-Mosel formula has been proven successfully in explaining the experimental results of fusion cross section. In this study, the fusion cross sections of $^{16}\mathrm{O} + {}^{16}\mathrm{O}$ reaction were calculated by using modified Glas-Mosel formula. The energies were set at $10 \leq E \leq 40$ MeV. The potential of interacting nuclei was approached by using Woods-Saxon potential. In numerical process, the differential equations were solved by using finite different method and optimization process was performed by using Nelder-Mead method. Good agreement between the experimental and this study results has been achieved successfully. Referring those results above, it can be indicated that the modified Glas-Mosel formula has good capability to explain the experimental results of fusion reaction of light nuclei. It can be a useful tool in explaining the experimental results or in predicting fusion cross section of light nuclei.

Keywords: Glas-Mosel formula, fusion reaction, cross section, light nuclei.

INTRODUCTION

One of the interesting topics that should be concerned in order to study about nuclear reaction, especially in charged-particle nuclear reactions, is the height of the barrier between the interacting nuclei is. It can provide information on the fusion process by a measurement. It is an important intermediate step to produce super-heavy nuclei by heavy-ion reactions.

In 1974, Glas and Mosel [1] proposed a model to analyze both high and low energies fusion cross section data for heavy-ion collisions. It has been proven successfully in explaining the experimental results by other researchers [2,3].

In this study, it was calculated the fusion cross section of $^{16}\mathrm{O} + {}^{16}\mathrm{O}$ reaction at $10 \leq E \leq 40$ MeV of energies by using modified Glas-Mosel formula. The calculation results have been also compared with the experimental results obtained by Fernandez *et al.* [4]. In addition, the calculation results of $^{16}\mathrm{O} + {}^{16}\mathrm{O}$ reaction obtained by the previous researchers, with different methods, have been also displayed. Simenel *et al.* [5] worked with the density-constrained time-dependent Hartree-Fock (DC-TDHF) method to explain the experimental results. Kondo *et al.* [6] used the optical model with the deep ion-ion potential to calculate the fusion cross sections.

Another purpose of this study is to observe the capability of modified Glas-Mosel formula and its precision in predicting the fusion cross section. It is very important information as a fundamental reference for further study about fusion cross section using heavy and super-heavy nuclei. It is also very useful to provide fusion cross section data, especially for unavailable experiment data or the difficulty in cross section measurement by experiments.

METHOD

In this study, it is considered first in calculation fusion cross sections that the interacting nuclei are spherical and without dynamical distortion. The various barriers are approximated for different partial waves by inverted harmonic-oscillator potentials of height E_l and frequency ω_l. For an energy E, the probability $P(l,E)$ for the absorption of the l-th partial wave is then given by the Hill-Wheeler formula [7]

$$P_{E,l} = \left\{1 + \exp\left(\frac{2\pi(V_B - E)}{\hbar\omega_l}\right)\right\}^{-1} \quad (1)$$

where $\hbar$ is the reduced Planck constant and V_B represents the height of barrier potential, well known as Coulomb barrier (MeV). The total potential can be calculated by using [8, 9]

$$V_{tot}(r) = \frac{Z_p Z_T e^2}{r} + \frac{-V_0}{\left[1 + \exp\left(\frac{r - R_0}{a}\right)\right]} \quad (2)$$

where Z represents the atomic number, e is the electron charge, V_0 represents the potential depth, r respresents the distance between the interacting nuclei, R_0 represents the radius, and a represents the surface diffuseness parameter. The parameters of V_0, R_p, and R_T can be estimated by using the following approximations [8, 9, 10]

$$V_0 = 16\pi\gamma\bar{R}a \quad (3)$$

$$R_0 = r_0\left(A_p^{1/3} + A_T^{1/3}\right) \quad (4)$$

$$R_c = r_c\left(A_p^{1/3} + A_T^{1/3}\right) \quad (5)$$

$$V_c(R_c) = \frac{Z_p Z_T e^2}{R_c} \tag{6}$$

$$R_{p(T)} = 1.23A_{p(T)}^{1/3} - 0.98A_{p(T)}^{-1/3} \tag{7}$$

$$\bar{R} = \frac{R_T R_p}{R_T + R_p} \tag{8}$$

$$\gamma = \gamma_0 \left[1 - k\left(\frac{(N_T - Z_T)(N_p - Z_p)}{A_T A_p}\right)\right] \tag{9}$$

where r_0 represents the radius parameter, R_c represents the critical distance, and r_c represents the parameter of critical distance. The constants of k and γ_0 were set respectively at 1.8 and 0.95 (in MeV·fm^{-2}) [8]. The subscript p and T refer to projectile and target respectively. The parameter of $\hbar\omega_l$ (where $\hbar\omega_l \simeq \hbar\omega_B$), represents the characterizing the behavior of the fusion cross section at very low energy near and below the Coulomb barrier, which can be approached by

$$\hbar\omega_B = \hbar\left[\frac{1}{\mu}\frac{d^2V(r)}{dr^2}\bigg|_{R_B}\right]^{1/2} \tag{10}$$

where μ is the reduced mass of nuclei. The parameter $\hbar\omega$ is the (l-independent) curvature of the parabolic barrier assumed in the calculation. Finally, the fusion cross sections can be calculated by using Wong formula [7, 9]

$$\sigma_F(E) = \frac{\hbar\omega_B R_B^2}{2E}\ln\left\{1 + \exp\left(\frac{2\pi(E - V_B)}{\hbar\omega_B}\right)\right\} \tag{11}$$

In other hand, in the phenomenological model of Glas and Mosel, the fusion cross section is given by [1]

$$\sigma_F(E) = \frac{\hbar\omega R_B^2}{2E}\ln\left\{\frac{1 + \exp[2\pi(E - V_B)/\hbar\omega]}{1 + \exp\{2\pi[E - V_B - (E - V_c)R_c^2/R_B^2]/\hbar\omega\}}\right\} \tag{12}$$

where R_B (R_c) represents the barrier (critical) distance and V_c represents the critical potential. The parameter V_B is obtained as the peak of total potential.

In this study, it is proposed a modified Glas-Mosel formula as

$$\sigma_F(E) = \frac{\hbar\omega R_B^2}{2E}\ln\left\{\frac{1 + \exp[2\pi(E - V_B)/\hbar\omega]}{1 + \exp\{2\pi[C_1(E - V_c)R_c^2/R_B^2]/\hbar\omega\}}\right\} \tag{13}$$

where C_1 is the parameter. The calculations were performed numerically. The numerical differential was calculated by using finite different method, which can be formulated as [11]

$$\frac{d^2y}{dx^2} = \frac{y(x + \Delta x) - 2y + y(x - \Delta x)}{(\Delta x)^2} \tag{14}$$

where Δx represents the step of axis. For optimizing the parameters of a, r_0, r_c, and C_1, it was utilized Nelder-Mead method. This method is very useful to optimize computational problems by using numerical method or to solve the analytic problems with unknown gradient [12]. The capability of this method has been proven successfully in solving the function with two variation parameters [12, 13].

The step of this method is not so complicated. For example, consider $f(x, y)$ as function. The first step, it is taken three trial coordinates, i.e. (x_1, y_1), (x_2, y_2), and (x_3, y_3). After calculated, those coordinats are noted as follows: $B = (x_1, y_1)$ as the best vertex (vertex with minimum value), $G = (x_1, y_1)$ as good vertex, dan $W = (x_1, y_1)$ as the worst vertex. Next step, the function is optimized by following the algorithm explained in Table 1. Many examples in optimizing the functions can be found in the following reference [12].

Table 1. The algorithm of Nelder-Mead method. [12]

<table>
<tr><td colspan="2">Case (i)
Determine vertex as B, G, and W
Compute
$M = (B + G)/2$, $R = 2M - W$, and $E = 2R - M$
If $f(R) < f(G)$, then
Perform case (ii) → either reflect or extend
Else
Perform case (iii) → either contract or shrink</td></tr>
<tr><td>Case (ii)
Begin
If $f(B) < f(R)$ then
Replace W with R
Else
Compute E and $f(E)$
If $f(E) < f(B)$ then
Replace W with E
Else
Replace W with R
End if

End if
End Case (ii)</td><td>Case (iii)
Begin
If $f(B) < f(W)$ then
Replace W with R
Compute $C = (W + M)/2$
or $C = (R + M)/2$ and $f(C)$
If $f(C) < f(W)$ then
Replace W with C
Else
Compute S and $f(S)$
Replace W with S
Replace G with M
End if
End Case (iii)</td></tr>
</table>

To ensure the precision of calculation, chi-square distribution can be utilized as long as the experimental data can be extracted. The chi-square can be calculated by using [14,15]

$$\chi^2 = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\left(\sigma_i^{\text{theory}} - \sigma_i^{\text{exp}}\right)}{\Delta\sigma_i^{\text{exp}}}\right]^2 \tag{15}$$

where $\Delta\sigma_i^{\text{exp}}$ represents the uncertainty in measurements and N is the number of data. The minimum value of χ^2 measures the quality of the fit. The smaller the value of χ^2, the higher the quality of the fit [14].

RESULTS AND DISCUSSION

Calculation process in this study is commenced by optimizing a, r_0, r_c, and C_1 in Nelder-Mead framework. These parameters were optimized in order to get minimum chi-square of fusion cross section. To guarantee the saturation of calculation, the chi-square tolerance was set at 10^{-20}. The obtained results of this process are: $a = 0.6282$ fm, $r_0 = 1.2072$ fm, $r_c = 1.1822$ fm, and $C_1 = 0.7012$ with $\chi^2 = 0.4106$. It can be seen that the obtained chi-square is less than 0.5 which means that the excellent agrement between this calculation and the experimental result has been achieved successfully.

Next step after optimization, it is calculated the fusion cross sections for $^{16}\mathrm{O} + {}^{16}\mathrm{O}$ reaction by using the modified Glas-Mosel formula, as shown in eq. (13). The calculation results of some parameters obtained inthis study are shown in Table 2. The potential models used in this study have been displayed in Figure 1. It can be seen from Figure 1 that, at distance above 6 fm, Woods-Saxon potential moves in approaching zero exponentially. The Coulomb potential also approaches to zero smoothly.

Table 2. Calculation results of $^{16}\mathbf{O} + {}^{16}\mathbf{O}$ reaction.

Parameter	The obtained result	Parameter	The obtained result
R_p (fm)	2.71049	V_B (MeV)	9.994677
R_T (fm)	2.71049	R_B (fm)	8.5
$\bar{R}$ (fm)	1.355246	V_c (MeV)	15.472342
V_0 (MeV)	40.654592	R_c (fm)	5.957915
R_0 (fm)	6.083907	$\hbar\omega_B$ (MeV)	2.991378

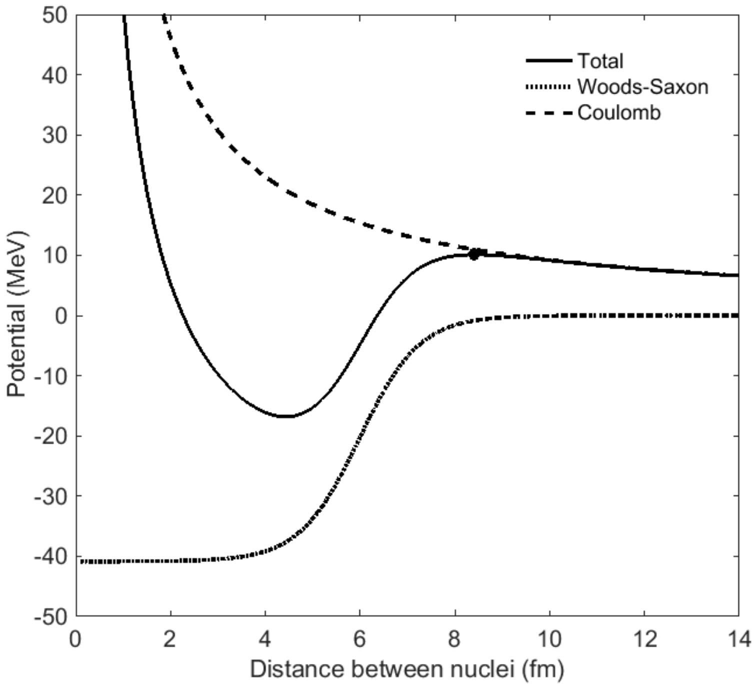
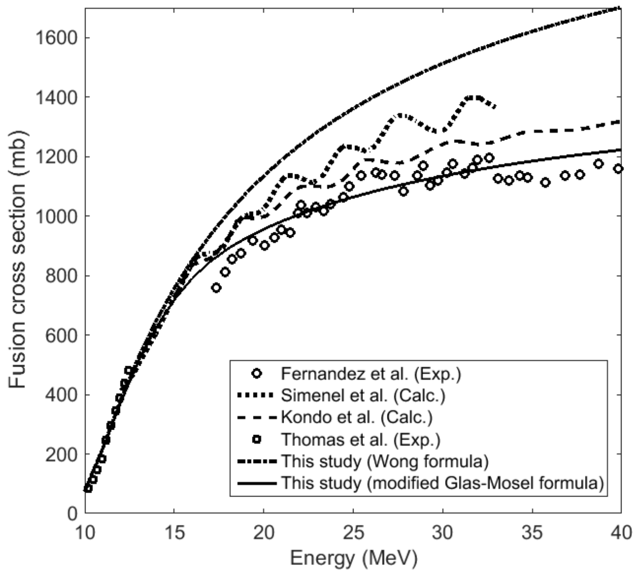
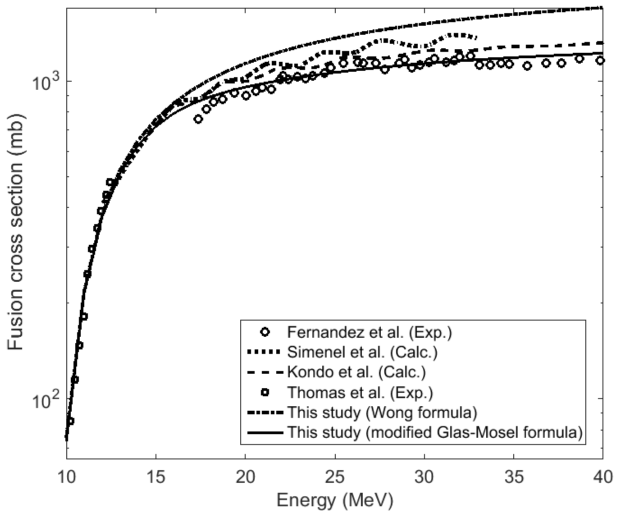


Figure 1. The potential model for $^{16}\mathrm{O} + {}^{16}\mathrm{O}$ reaction.

Figure 2 shows the calculation results of fusion cross sections obtained in this study. It has been also compared to the experimental results obtained by Fernandez *et al*. and to the calculation results of Simenel *et al*. and Kondo *et al.* As comparison, it is also calculated the fusion cross section by using Wong formula as shown in eq. (11). The results of this study by using Wong formula are higher than those of Simenel *et al.* By using the modified Glas-Mosel formula, the results of this study are almost similar to those of experiment.



(a) linear scale



(b) logarithmic scale

Figure 2. The fusion cross sections for $^{16}O + ^{16}O$, obtained by Fernandez *et al.* [4], Simenel *et al.* [5], Kondo *et al.* [6], Thomas *et al.* [16], and this study.

The modification of Glas-Mosel formula proposed in this study has reduced the fusion cross section value obtained by using Wong formula, especially at $15 \leq E \leq 40$ MeV of energies. This study results is also close to the calculation results of Kondo *et al*. It means that, in using the modified Glas-Mosel formula, when the condition of $E \geq V_c$ is achieved, the fusion cross section values decrease. This decrease is influenced by the denominator term of log function of eq. (13). Therefore, good agreement between this study results and the experimental results has been achieved successfully for $^{16}O + ^{16}O$ reaction.

Based on those results above, the modified Glas-Mosel formula, with the obtained parameters, proposed in this study, has good capability in explaining the experimental results of fusion cross sections of $^{16}O + ^{16}O$. It can be inferred that the modified Glas-Mosel formula can be a useful tool in explaining the experimental results or in predicting fusion cross section of light nuclei in order to provide the nuclear data. In other hand, it should be applied to the heavy nuclei, even super-heavy nuclei, to investigate the capability of this formula.

SUMMARY

It has been calculated the fusion cross sections for $^{16}\mathrm{O} + {}^{16}\mathrm{O}$ by using modified Glas-Mosel formula at $10 \leq E \leq 40$ MeV of energies. The potential of the interacting nuclei was approached by using Woods-Saxon potential. The calculations were performed numerically by using finite different and Nelder-Mead methods. An excellent agreement with the experimental results has been achieved successfully in this study. Referring those results above, it can be indicated that the modified Glas-Mosel formula has good capability to explain the experimental results of fusion reaction of light nuclei. It can be a useful tool in explaining the experimental results or in predicting fusion cross section of light nuclei.

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