p-ISSN 2089 – 0133 e-ISSN 2477-6416
URL : https://jurnal.uns.ac.id/ijap/article/view/101537

DOI : https://doi.org/10.13057/ijap.v16i1.101537

ANALYTICAL SOLUTION OF THE GENERALIZED FRACTIONAL NEWTON'S COOLING LAW BY GENERALIZED FRACTIONAL LAPLACE TRANSFORM METHOD

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Received 2025-04-28, Revised 2026-01-27, Accepted 2026-06-03,
Available Online, Published Regularly June 2026

ABSTRACT

This study presents an analytical framework for a generalized fractional formulation of Newton's law of cooling based on a newly defined generalized fractional (GF) Laplace transform. The proposed formulation extends existing fractional models by incorporating both a fractional order and a non-negative continuous kernel function, thereby providing increased modelling flexibility and a more general representation of memory-dependent thermal processes.
A rigorous definition of the GF Laplace transform is established, together with its fundamental properties, including linearity, existence conditions, and operational rules for generalized fractional derivatives and integrals. Within this framework, the classical Newton's cooling equation is reformulated in terms of the GF derivative, and an explicit analytical solution is obtained using the proposed transform method. The resulting solution generalizes the classical exponential decay law and reduces to the standard model under appropriate parameter selection.
In addition, the effects of the fractional order and kernel function on cooling dynamics were examined through several representative cases. The analysis showed that both parameters played a significant role in determining the rate of temperature decay. In particular, smaller fractional orders lead to stronger memory effects and slower convergence, whereas the choice of the kernel function determines the effective time scale of the system. These results demonstrate the capability of the generalized fractional framework to describe a wider range of thermal behaviors than classical and existing fractional models.

Keywords: Generalized fractional (GF); GF Laplace transform; GF Newton's cooling law

INTRODUCTION

Fractional calculus extends classical calculus by allowing derivatives and integrals of non-integer orders, and has found wide-ranging applications in physics, engineering, and chemistry. Numerous definitions have been proposed, including the Riemann–Liouville, Caputo, Hadamard, and Weyl formulations. However, most of these approaches do not fully preserve the fundamental properties of classical calculus, such as the chain rule, product rule, and key theorems, including Rolle's theorem and the mean value theorem [1]. In addition, their integral-based structures often make it challenging to derive analytical solutions for complex models [2].

As an alternative, Khalil *et al.* [3] introduced the conformable fractional (CF) calculus, which retains several essential properties of the classical calculus while offering a more straightforward analytical framework. Despite these advantages, CF calculus remains limited in terms of modelling flexibility, as it depends solely on a single parameter, namely, the fractional order. To address this limitation, several generalizations have been developed, including the new fractional (NF), local fractional (LF), new conformable fractional (NCF), and generalized conformable fractional (GCF) calculus [4–7].

Akkurt *et al.* [8] proposed a further advancement using^[8] generalized fractional (GF) calculus, which unifies several earlier approaches within a broader framework. Unlike conventional fractional models, the GF calculus incorporates two parameters: the fractional order and a non-negative continuous kernel function. This structure enhances the modelling flexibility and allows for a more comprehensive description of the dynamic processes. Nevertheless, its utilization, particularly in conjunction with analytical solution techniques, remains relatively underexplored.

In contrast, the Laplace transform is a fundamental tool for solving differential equations, including those arising in the fractional calculus. Although various fractional Laplace transform formulations have been developed [1,7,9–11], approaches that are fully compatible with the GF framework, especially those involving general kernel functions, have not yet been systematically investigated.

This gap becomes more pronounced in applications to physical models, such as Newton's law of cooling. Fractional formulations have been shown to provide more accurate representations of experimental data than classical models [12]. Although several studies have examined the Newton cooling model within the CF framework [12–16], a flexible GF based formulation combined with a consistent Laplace transform approach has not been thoroughly developed.

Motivated by these gaps, this study proposes a new definition of the Laplace transform within the generalized fractional framework, constructed based on a non-negative continuous function q and an invertible function φ, along with an investigation of its main properties. Furthermore, Newton's law of cooling is reformulated within the GF setting and solved using the proposed transformation. The main contribution of this study lies in integrating the flexibility of the GF structure with a consistent Laplace transform method, resulting in analytical solutions that are more general and better suited to capturing diverse thermal system behaviors.

THE GF CALCULUS

In this section, the definitions of the GF derivative and corresponding GF integral are introduced. In addition, several fundamental properties, as reported in [8], are presented to support the subsequent analysis.

Definition 1. Let $q\colon [0,\infty) \to R$ be a continuous, nonnegative function such that $q(a) = 0$, with $q(t) \neq 0$ and $q'(t) \neq 0$ for all $t > a$. Let $f\colon [0,\infty) \to R$ be a real-valued function and let $\eta \in (0,1]$. Then, the GF-derivative of f with respect to q of order η, denoted by $D_q^{\eta} f(t)$, is defined by

$$D_q^{\eta} f(t) = \frac{f\left(t - q(t) + q(t) expexp\left(\varepsilon q(t)^{-\eta} q'(t)\right)\right) - f(t)}{\varepsilon}. \tag{1}$$

From this point onward, the function q is assumed to satisfy the conditions specified in Definition 1.

Remarks 1.

1. When $q(t) = t$, Equation (1) reduces to the derivative introduced by Katugampola [4].
2. By setting $\eta = 1$ and $q(t) = t$ in Equation (1), the definition simplifies to the classical derivative.

Theorem 1. Let $\eta \in (0,1]$, and let $f: [0, \infty) \to R$ be a continuous function that is GF differentiable for all $t > a$. Then, the function f is GF-differentiable with respect to q of order η at the point t, and

$$D_q^\eta f(t) = \frac{\left(q(t)\right)^{1-\eta}}{q'(t)} f'(t). \tag{2}$$

The fundamental properties of the GF derivative for certain classes of functions are as follows.

Theorem 2. Let $\eta \in (0,1]$, and suppose that $f$and g are GF differentiable functions with respect to q for all $t \in (0, \infty)$. Then, the following properties hold on the interval $[0, \infty)$:

1. Linearity. The function $kf + lg$ is GF-differentiable, and
$$D_q^\eta(kf(t) + lg(t)) = k\, D_q^\eta f(t) + l\, D_q^\eta g(t),$$
for *all* $k, l \in R$.
2. Product Rule. The function fg is GF-differentiable, and
$$D_q^\eta(fg)(t) = f(t)\, D_q^\eta g(t) + g(t)\, D_q^\eta f(t).$$
3. Quotient Rule. The function f/g is GF-differentiable, provided that $g(t) \neq 0$, and
$$D_q^\eta \left(\frac{f}{g}\right)(t) = \frac{g(t)\, D_q^\eta f(t) - f(t)\, D_q^\eta g(t)}{(g(t))^2}.$$
4. Chain Rule. The composition $f \circ g$ is GF-differentiable, and
$$D_q^\eta(f \circ g)(t) = \left(q(t)\right)^{1-\eta} q'(t)\, f'(g(t))\, g'(t).$$

We now introduce the definition of a generalized fractional integral.

Definition 2. Let $\eta \in (0,1]$, and let $f$be defined on the interval $(0, t]$. If $f$is GF integrable with respect to q for every $t > 0$, then the GF integral of $f$with respect to q of order η, denoted by $I_q^\eta f(t)$, is defined as

$$I_q^\eta f(t) = \int_0^t f(x)\, d(q(x)^\eta) = \int_0^t f(x)\, \left(q(x)\right)^{\eta-1} q'(x)\, dx, \tag{3}$$

provided that the Riemann integral exists.

Next, the generalized fractional integration by-parts theorem is presented.

Theorem 3. Let $\eta \in (0,1]$, and suppose that $f, g: [a, b] \to R$ are GF differentiable functions of order η with respect to q. Then, the following relation is obtained:

$$\int_a^b f(t)\, D_q^\eta g(t)\, d\left(\frac{(q(t))^\eta}{\eta}\right) = f(t)g(t)\Big|_a^b - \int_a^b g(t)\, D_q^\eta f(t)\, d\left(\frac{(q(t))^\eta}{\eta}\right). \tag{4}$$

NEWTON'S COOLING LAW EQUATION

Newton's law of cooling is a first-order ordinary differential equation that describes the temporal evolution of the temperature of an object. This law states that the rate at which the temperature changes is proportional to the temperature difference between the object and its surrounding environment. As presented in [12], the model can be written as

$$\frac{dS}{dt} = -\lambda(S(t) - S_a), S(0) = S_0,\ t \in (0, \infty) \tag{5}$$

Here, $S(t)$denotes the temperature of the object at time t, $S_a$represents the ambient temperature, S_0 is the initial temperature at $t = 0$, and λ is the cooling coefficient. In this study, the ambient temperature S_a is assumed to remain constant, while the parameter $\lambda > 0$ has the dimension of inverse time and characterizes the heat transfer rate between the object and its surroundings. The analytical solution of Equation (5) is given by

$$S(t) = S_a + (S_0 - S_a)e^{\lambda t} \tag{6}$$

The coefficient λ has units of inverse time s^{-1}. Equation (5) describes the evolution of the temperature difference between the object's initial temperature S_0 and the ambient temperature S_a. Suppose that after a given time interval τ, the temperature changes from $S(t)$ to $S(t_1) = S_1$. Under these conditions, the parameter λ can be determined from Eq. (6),

$$\lambda = \frac{1}{\tau} ln\ ln \left(\frac{S_0 - S_a}{S_1 - S_a}\right). \tag{7}$$

This expression indicates that $S(t)$approaches S_0 as $t \to 0$, and converges to $S_a$as $t \to \infty$. Consequently, the temperature of the object gradually approaches the ambient temperature, leading the system toward thermal equilibrium.

THE GF NEWTON'S COOLING LAW EQUATION

In this section, Newton's law of cooling is reformulated within the framework of generalized conformable fractional calculus. This approach yields a fractional extension of the classical model by incorporating the GF derivative operator, thereby allowing memory effects to be captured more effectively

To maintain consistency in the fractional representation of the model, the classical first-order time derivative is replaced with a fractional-order operator in the sense of the Caputo fractional derivative, following the approach in [12]. The modified differential operator is defined by

$$\frac{d}{dt} = \lambda^{1-\eta} \frac{d^\eta}{dt^\eta} \tag{8}$$

where $\frac{d^\eta}{dt^\eta}$ is the Caputo fractional derivative of order $\eta \in (0,1]$. By substituting Eq. (8) into the classical cooling equation, the following fractional model is obtained:

$$\frac{d^\eta S}{dt^\eta} = -\lambda^\eta(S(t) - S_a), S(0) = S_0, t \in (0, \infty), \eta \in (0,1]. \tag{9}$$

Furthermore, a generalized formulation is derived by replacing the Caputo fractional derivative in Equation (9) with the GF derivative operator D_q^η. This leads to the model

$$D_q^\eta S(t) = -\lambda^\eta(S(t) - S_a), S(0) = S_0, t \in (0, \infty), \eta \in (0,1]. \tag{10}$$

This generalized formulation extends the classical cooling law into the fractional domain while preserving its essential physical interpretation, thereby providing a more flexible framework for describing heat transfer processes with nonlocal or memory-dependent effects.

THE GF LAPLACE TRANSFORM METHOD

Inspired by earlier developments of fractional Laplace transforms [7],[9],[11], the generalized fractional Laplace transform is defined herein, and its essential properties are established to support the analytical framework of this study.

Definition 3. *Let $\eta \in (0,1]$, and let $f: [0,\infty) \to R$ be a given function. A function is said to be GF q -exponentially bounded of order $c \in R$ if there exist constants $H > 0$ and c such that*

$$| f(t) | \leq H \, exp \, exp \left(c \frac{(q(t))^{\eta}}{\eta}\right),$$

for all $t \geq 0$.

We consider the following Laplace-type integral transform in the sense of Ansari [11]

$L_q[f(t); s] = \int_0^\infty \; q'(t) exp(-\varphi(s) q(t)) f(t) \, dt,$ where $s \in C$ is a complex parameter, $\varphi: C \to C$ is an analytic and invertible function, and q is a continuously differentiable function.

Motivated by the above Laplace-type transform of Ansari, we now introduce a generalized fractional Laplace transform constructed through a nonnegative continuous function q and an invertible function φ. This generalized framework extends the classical transform by incorporating a fractional-order structure associated with the parameter $\eta \in (0,1]$.

Definition 4. *Let $\eta \in (0,1]$, let $f: [0,\infty) \to R$ be continuous (or piecewise continuous), and let $\varphi: C \to C$ be an analytic and invertible function on a suitable domain. The GF Laplace transform of f with respect to the function q of order η, denoted by $L^{\eta}_{q,\varphi}[f(t), s]$, is defined by*

$$L^{\eta}_{q,\varphi}[f(t); s] = F(s) = \int_0^\infty \; exp\left(-\varphi(s) \frac{(q(t))^{\eta}}{\eta}\right) f(t) \, (q(t))^{\eta-1} q'(t) \, dt,$$

for all $s \in C$ such that $Re(\varphi(s)) > c$ provided that the integral exists.

Equivalently, using the substitution

$$d\left(\frac{(q(t))^{\eta}}{\eta}\right) = (q(t))^{\eta-1} q'(t),$$

the transform can be written as

$$F(s) = \int_0^\infty \; f(t) \; d\left(\frac{(q(t))^{\eta}}{\eta}\right), Re(\varphi(s)) > c. \quad (11)$$

To complete the framework of the generalized fractional Laplace transform, we now introduce its inverse operator, which allows the recovery of the original function from its transform representation.

Definition 5. *Let $\eta \in (0,1]$, and let $F(s)$ be the GF Laplace transform of a function f. Assume that:*

1. *$f: [0,\infty) \to R$ is continuous (or piecewise continuous),*

2. *f is GF q −exponentially bounded, i.e., there exist constants* $H > 0$ *and* $c \in R$ *such that*

$$\mid f(t) \mid \leq H\, exp\, exp\left(c\frac{(q(t))^{\eta}}{\eta}\right),$$

φ is an analytic and invertible function in a vertical strip of the complex plane containing the contour of integration.

Then the inverse GF Laplace transform of F, denoted by $\left(L_{q,\varphi}^{\eta}\right)^{-1}[F(s);t]$, *is defined by*

$$f(t) = \left(L_{q,\varphi}^{\eta}\right)^{-1}[F(s);t]$$
$$= \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} exp\, exp\left(\varphi(s)\frac{(q(t))^{\eta}}{\eta}\right) F(s)\, d(s) \quad (12)$$

where the integration is taken along the vertical line $Re(\varphi(s)) = \omega > c$ *and the integral is assumed to converge.*

Remarks 2. The following integral transforms are special cases of the GF Laplace transform.

1. If $\eta = 1$, then the GF Laplace transform reduces to the Ansari sense Laplace transform [11],

$$L_{q,\varphi}^{1}[f(t);s] = F(s) = \int_0^{\infty} q'(t)\ exp\, exp\,(-\varphi(s)\, q(t))\, f(t)\, dt$$

2. If $q(s) = -s$, $q(t) = ln(t)$, and $\eta = 1$, then the GF Laplace transform reduces to the Mellin transform [17],

$$L_{ln(t),-s}^{1}[f(t);s] = F(s) = \int_0^{\infty} t^{s-1} f(t)\, dt$$

3. If $\varphi(s) = s$, $q(t) = t$, and $\eta \in (0,1)$, then the GF Laplace transform reduces to the conformable Laplace transform as presented in [1]

$$L_{t,s}^{\eta}[f(t);s] = F(s) = \int_0^{\infty} exp\, exp\left(-s\frac{t^{\eta}}{\eta}\right) f(t)\, dt.$$

4. If $\varphi(s) = s$, $q(t) = t$, and $\eta = 1$, then the GF Laplace transform reduces to the classical (standard) Laplace transform,

$$L_{t,s}^{1}[f(t);s] = F(s) = \int_0^{\infty} exp\, exp\,(-st)\, f(t)\, dt.$$

5. If $\varphi(s) = s^2$, $q(t) = t^2$, and $\eta = 1$, then the GF Laplace transform reduces to the $L_{s,t}^{1}$ transform [18-19],

$$L_{t^2,s^2}^{1}[f(t);s] = F(s) = \int_0^{\infty} 2s\, exp\, exp\,(-s^2t^2)\ f(t)\, dt.$$

The following theorem provides a sufficient condition for the existence of the GF Laplace transform.

Theorem 4. Let $\eta \in (0,1]$, and let $\varphi: C \to C$ be analytic and invertible on a domain $D \subset C$. If the function $f: [0,\infty) \to R$ is continuous (or piecewise continuous) and GF q-exponentially bounded, then the GF Laplace transform $L_{q,\varphi}^{\eta}[f(t);s], t \geq 0$ exists for all $s \in D$ such that $Re(\varphi(s)) > c$.

Proof. By Definition 4, the GF Laplace transform of f is given by

$$L_{q,\varphi}^{\eta}[f(t);s]=\int_0^{\infty} \quad exp\,exp\left(-\varphi(s)\frac{(q(t))^{\eta}}{\eta}\right)f(t)\,d\left(\frac{(q(t))^{\eta}}{\eta}\right).$$

Taking the absolute value, we obtain

$$\mid L_{q,\varphi}^{\eta}[f(t);s]\mid \leq \int_0^{\infty} \quad exp\,exp\left(-Re(\varphi(s))\frac{(q(t))^{\eta}}{\eta}\right)\mid f(t)\mid\, d\left(\frac{(q(t))^{\eta}}{\eta}\right).$$

Since f is GF q-exponentially bounded, there exist positive constants H and $c\in R$ such that

$$\mid f(t)\mid\leq H\,exp\,exp\left(c\frac{(q(t))^{\eta}}{\eta}\right).$$

Hence,

$$\mid L_{q,\varphi}^{\eta}[f(t);s]\mid \leq H\int_0^{\infty} \quad exp\,exp\left(-(Re(\varphi(s))-c)\frac{(q(t))^{\eta}}{\eta}\right)d\left(\frac{(q(t))^{\eta}}{\eta}\right).$$

Evaluating the integral yields

$$\mid L_{q,\varphi}^{\eta}[f(t);s]\mid\leq\frac{H}{Re(\varphi(s))-c}, Re(\varphi(s))>c.$$

This completes the proof.

□

The following theorem shows that Definition 4 satisfies the linearity property.

Theorem 5. Let $\eta\in(0,1]$, and let $\varphi: C\to C$ be analytic and invertible on a domain $D\subset C$. Suppose that $f,g:[0,\infty)\to R$ are functions such that their GF Laplace transforms $L_{q,\varphi}^{\eta}[f(t);s], t\geq 0$ and $L_{q,\varphi}^{\eta}[g(t);s], t\geq 0$ exist for all $s\in D$ such that . Then, for any constants $k,l\in R$, we have

$$L_{q,\varphi}^{\eta}[kf(t)+lg(t);s]=k\,L_{q,\varphi}^{\eta}[f(t);s]+l\,L_{q,\varphi}^{\eta}[g(t);s], Re(\varphi(s))>c. \quad (13)$$

The relationship between the GF Laplace transform $L_{q,\varphi}^{\eta}$, and the classical Laplace transform $L_{q,t}^{1}$ is established in the following theorems.

Theorem 6. Let $\eta\in(0,1]$, and let $\varphi: C\to C$ be analytic and invertible on a domain $D\subset C$. Suppose that $f:[0,\infty)\to R$ is such that both $L_{q,\varphi}^{\eta}[f(t);s]$ and $L_{q,\varphi}^{1}[f(t);s]$ exist. Then, for all $s\in D$ such that $Re(\varphi(s))>c$, the following identity holds:

$$L_{q,\varphi}^{\eta}\left[f\left(\frac{(q(t))^{\eta}}{\eta}\right);s\right]=L_{q,t}^{1}[f(t);s]. \quad (14)$$

Proof. Let $u=\frac{(q(t))^{\eta}}{\eta}$, so that $du=d\left(\frac{(q(t))^{\eta}}{\eta}\right)$. Using this substitution in Equation (14), we obtain

$$L_{q,\varphi}^{\eta}\left[f\left(\frac{(q(t))^{\eta}}{\eta}\right);s\right]=\int_0^{\infty} \quad exp\,exp\left(-\varphi(s)\frac{(q(t))^{\eta}}{\eta}\right)f\left(\frac{(q(t))^{\eta}}{\eta}\right)d\left(\frac{(q(t))^{\eta}}{\eta}\right)$$

$$=\int_0^{\infty} \quad exp\,exp\,(-\varphi(s)u)\,f(u)\,du=L_{q,t}^{1}[f(t);s].$$

This completes the proof. □

Theorem 7. Let $\eta \in (0,1]$, and let $\varphi: C \to C$ be analytic and invertible on a domain $D \subset C$. Suppose that $f: [0, \infty) \to R$ is such that both $L^{\eta}_{q,\varphi}[f(t); s]$ and $L^{1}_{q,\varphi}[f(t); s]$. Define the function $g: [0, \infty) \to R$ by $g(u) = f\left(q^{-1}\left(\eta^{\frac{1}{\eta}} u^{\frac{1}{\eta}}\right)\right)$. Then, for all $s \in D$ such that $Re(\varphi(s)) > c$, we have

$$L^{1}_{q,\varphi}[g(u); s] = L^{\eta}_{q,\varphi}[f(t); s].$$

Proof. Assume that $q$is a non-negative continuous function satisfying $q(0) = 0$ and $q(t), q'(t) \neq 0$. Under these conditions, the inverse function q^{-1} exists. Hence, we obtain

$$L^{1}_{q,\varphi}[g(u); s] = \int_0^{\infty} \quad exp\, exp\,(-\varphi(s)u)\, f\left(q^{-1}\left((\eta u)^{\frac{1}{\eta}}\right)\right) du. \qquad (15)$$

Suppose that $q^{-1}\left((\eta u)^{\frac{1}{\eta}}\right) = t$ and $u = \frac{(q(t))^{\eta}}{\eta}$ in Eq. (15), we get

$$L^{1}_{q,\varphi}[g(u); s] = \int_0^{\infty} \quad exp\, exp\left(-\varphi(s)\frac{(q(t))^{\eta}}{\eta}\right) f(t)\, d\left(\frac{q(t)^{\eta}}{\eta}\right)$$
$$= L^{\eta}_{q,\varphi}[f(t); s].$$

This completes the proof. □

The scaling property of the GF Laplace transform is stated as follows.

Theorem 8. Let $\eta \in (0,1]$, and let $\varphi: C \to C$ be an analytic, invertible, and homogeneous of degree $m \in N$ on a domain $D \subset C$, i.e., $\varphi(\lambda s) = \lambda^{m}\varphi(s), \forall \lambda > 0, s \in C$. Suppose that $f: [0, \infty) \to R$ is such that its GF Laplace transform$L^{\eta}_{q,\varphi}[f(t); s] = F(s)$ exists for s in a domain D with $Re(\varphi(s)) > c$. Then, for any $\lambda > 0$,

$$L^{\eta}_{q,\varphi}\left[f\left(\lambda^{m}\frac{(q(t))^{\eta}}{\eta}\right); s\right] = \frac{1}{\lambda^{m}} F\left(\frac{s}{\lambda}\right).$$

Proof. Suppose that $u = \lambda^{m}\frac{(q(t))^{\eta}}{\eta}$ and $du = \lambda^{m}\, d\left(\frac{(q(t))^{\eta}}{\eta}\right)$. Substituting this change of variables into Eq. (11), we obtain

$$L^{\eta}_{q,\varphi}\left[f\left(\lambda^{m}\frac{(q(t))^{\eta}}{\eta}\right); s\right] = \frac{1}{\lambda^{m}}\int_0^{\infty} \quad exp\, exp\left(-\left(\frac{\varphi(s)}{\lambda^{m}}\right)u\right)\, f(u)\, du.$$

Since φ is homogeneous of degree m, we have $\varphi(\frac{s}{\lambda}) = \frac{\varphi(s)}{\lambda^{m}}$. Hence,

$$L^{\eta}_{q,\varphi}\left[f\left(\lambda^{m}\frac{(q(t))^{\eta}}{\eta}\right); s\right] = \frac{1}{\lambda^{m}}\int_0^{\infty} \quad exp\, exp\left(-\varphi\left(\frac{s}{\lambda}\right)u\right)\, f(u)\, du = \frac{1}{\lambda^{m}} F\left(\frac{s}{\lambda}\right).$$

This completes the proof. □

The following theorem presents the GF Laplace transform of the GF derivatives.

Theorem 9. Let $\eta \in (0,1]$, and let $f: [0, \infty) \to R$ be a sufficiently smooth function such that its GF Laplace transform $L^{\eta}_{q,\varphi}[f(t); s] = F(s)$ exists for s in a domain D with $Re(\varphi(s)) > c$.

Assume further that the GF derivatives $D_q^\eta f$ and $(D_q^\eta)^2 f$ are piecewise continuous and of GF q -exponential order. Then the following identities hold:

$$L_{q,\varphi}^{\eta}\left[D_q^\eta f(t); s\right] = \varphi(s)F(s) - f(0),$$

and

$$L_{q,\varphi}^{\eta}[(D_q^\eta)^2 f(t); s] = (\varphi(s))^2 F(s) - \varphi(s)f(0) - D_q^\eta f(0).$$

Proof. By Definition 4,

$$L_{q,\varphi}^{\eta}\left[D_q^\eta f(t); s\right] = \int_0^\infty \quad exp\, exp\left(-\varphi(s)\frac{(q(t))^\eta}{\eta}\right) D_q^\eta f(t)\, d\left(\frac{q(t)^\eta}{\eta}\right)$$

Applying integration by parts with

$$u = exp\, exp\left(-\varphi(s)\frac{(q(t))^\eta}{\eta}\right), dv = D_q^\eta f(t)\, d\left(\frac{q(t)^\eta}{\eta}\right),$$

we obtain

$$L_{q,\varphi}^{\eta}\left[D_q^\eta f(t); s\right] = f(t)\, exp\, exp\left(-\varphi(s)\frac{(q(t))^\eta}{\eta}\right)\Big|_0^\omega$$

$$+\varphi(s)\int_0^\infty \quad exp\, exp\left(-\varphi(s)\frac{(q(t))^\eta}{\eta}\right) f(t)\, d(q(t))^\eta.$$

Since f is of GF q —-exponential order and $Re(\varphi(s)) > c$ the boundary term at infinity vanishes. Therefore,

$$f(t)\, exp\, exp\left(-\varphi(s)\frac{(q(t))^\eta}{\eta}\right)\Big|_0^\omega = -f(0).$$

Hence,

$$L_{q,\varphi}^{\eta}\left[D_q^\eta f(t); s\right] = \varphi(s)F(s) - f(0).$$

For the second derivative, applying the first identity to $D_q^\eta f$ yields

$$L_{q,\varphi}^{\eta}[(D_q^\eta)^2 f(t); s] = \varphi(s)L_{q,\varphi}^{\eta}\left[D_q^\eta f(t); s\right] - D_q^\eta f(0).$$

Substituting the previous result,

$$L_{q,\varphi}^{\eta}[(D_q^\eta)^2 f(t); s] = \varphi(s)\big(\varphi(s)F(s) - f(0)\big) - D_q^\eta f(0),$$

which simplifies to

$$L_{q,\varphi}^{\eta}[(D_q^\eta)^2 f(t); s] = (\varphi(s))^2 F(s) - \varphi(s)f(0) - D_q^\eta f(0).$$

This completes the proof. □

The ensuing theorem shows the GF Laplace transform for the GF integral.

Theorem 10. Let $\eta \in (0,1]$, and let $f: [0, \infty) \to R$ be a sufficiently smooth function such that its GF Laplace transform $L_{q,\varphi}^{\eta}[f(t); s] = F(s)$ exists for s in a domain D with $Re(\varphi(s)) > c$.

Assume that $I_q^\eta f$ is GF differentiable and that the hypotheses of Theorem 9 are satisfied. Then,

$$L_{q,\varphi}^{\eta}\left[\int_0^x \quad f(t)\, d\left(\frac{(q(t))^{\eta}}{\eta}\right); s\right] = \frac{F(s)}{\varphi(s)}.$$

Proof. Consider $L_{q,\varphi}^{\eta}\left[D_q^{\eta}\left(\int_0^x \quad f(t)\, d\left(\frac{(q(t))^{\eta}}{\eta}\right)\right); s\right]$. By the linearity of the GF Laplace transform and the definition of the GF derivative, we obtain

$$L_{q,\varphi}^{\eta}\left[D_q^{\eta}\left(\int_0^x \quad f(t)\, d\left(\frac{(q(t))^{\eta}}{\eta}\right)\right); s\right] = \varphi(s)\, L_{q,\varphi}^{\eta}\left[\int_0^x \quad f(t)\, d\left(\frac{(q(t))^{\eta}}{\eta}\right); s\right].$$

By the fundamental theorem of GF calculus, it follows that

$$D_q^{\eta}\left(\int_0^x \quad f(t)\, d\left(\frac{(q(t))^{\eta}}{\eta}\right)\right) = f(x).$$

Taking the GF Laplace transform of both sides yields

$$L_{q,\varphi}^{\eta}[f(t); s] = \varphi(s)\, L_{q,\varphi}^{\eta}\left[\int_0^x \quad f(t)\, d\left(\frac{(q(t))^{\eta}}{\eta}\right); s\right].$$

Hence,

$$L_{q,\varphi}^{\eta}\left[\int_0^x \quad f(t)\, d\left(\frac{(q(t))^{\eta}}{\eta}\right), s\right] = \frac{F(s)}{\varphi(s)}.$$

This completes the proof. □

The translation property of the GF Laplace transform is stated as follows.

Theorem 11. *Let $\eta \in (0,1]$, and let $\varphi: C \to C$ be an analytic, invertible, and translation function satisfying $\varphi(s) - \varphi(c) = \varphi(s-c)$, $s, c \in C$. Suppose that $f: [0,\infty) \to R$ is a function whose GF Laplace transform $L_{q,\varphi}^{\eta}[f(t); s] = F(s)$ exists for s in a domain D with $Re(\varphi(s)) > c$. Then,*

$$L_{q,\varphi}^{\eta}\left[exp\ exp\ \left(\varphi(c)\left(\frac{q(t)}{\eta}\right)^{\eta}\right) f(t)\,; s\right] = F(s-c).$$

Proof. By the definition of the GF Laplace transform, we have

$$L_{q,\varphi}^{\eta}\left[exp\ exp\ \left(\varphi(c)\frac{(q(t))^{\eta}}{\eta}\right) f(t); s\right] =$$

$$\int_0^{\infty} \quad exp\ exp\ \left(-\varphi(s)\frac{(q(t))^{\eta}}{\eta}\right)\ exp\ exp\ \left(\varphi(c)\frac{(q(t))^{\eta}}{\eta}\right) f(t)\, d\left(\frac{(q(t))^{\eta}}{\eta}\right)$$
$$= \int_0^{\infty} \quad exp\ exp\ \left(-(\varphi(s)-\varphi(c))\frac{(q(t))^{\eta}}{\eta}\right) f(t)\, d\left(\frac{(q(t))^{\eta}}{\eta}\right).$$

Since φ satisfies the translation property, we have $\varphi(s) - \varphi(c) = \varphi(s-c)$. Hence,

$$L_{q,\varphi}^{\eta}\left[exp\ exp\ \left(\varphi(c)\frac{(q(t))^{\eta}}{\eta}\right)\ f(t); s\right]$$

$$= \int_0^{\infty} \quad exp\ exp\ \left(-\varphi(s-c)\frac{(q(t))^{\eta}}{\eta}\right) f(t)\, d\left(\frac{(q(t))^{\eta}}{\eta}\right)$$

$$= F(s-c).$$

This completes the proof. □

The following examples of GF Laplace transform for certain functions are provided.

Example 1. Let $\eta \in (0,1]$, and assume that the corresponding GF Laplace transforms exist in the region $Re(\varphi(s)) > c$. The following formulas illustrate some elementary GF Laplace transforms.

1. $L_{q,\varphi}^{\eta}[1; s] = \frac{1}{\varphi(s)}$,
2. $L_{q,\varphi}^{\eta}\left[\left(\frac{(q(t))^{\eta}}{\eta}\right)^{m}; s\right] = \frac{m!}{(\varphi(s))^{m+1}}$ where m is a positive integer,
3. $L_{q,\varphi}^{\eta}\left[exp\ exp\left(\varphi(c)\frac{(q(t))^{\eta}}{\eta}\right); s\right] = \frac{1}{\varphi(s-c)}$ where φ is a translation function,
4. $L_{q,\varphi}^{\eta}\left[sin\left(\varphi(a)\frac{(q(t))^{\eta}}{\eta}\right); s\right] = \frac{\varphi(a)}{(\varphi(s))^2+(\varphi(a))^2}$,
5. $L_{q,\varphi}^{\eta}\left[cos\left(\varphi(a)\frac{(q(t))^{\eta}}{\eta}\right); s\right] = \frac{\varphi(s)}{(\varphi(s))^2+(\varphi(a))^2}$.

ANALYTICAL SOLUTION OF GCF NEWTON'S COOLING LAW BY GCF LAPLACE TRANSFORM METHOD

The solution of the GF formulation of Newton's law of cooling is obtained by employing the GF Laplace transform. The derivation begins by applying the GF Laplace transform operator introduced in Definition 4 to the GF Newton's cooling equation given in Eq. (9). This yields

$$L_{q,\varphi}^{\eta}[D_q^{\eta}S(t); s] = L_{q,\varphi}^{\eta}[\lambda^{\eta}(S(t) - S_a); s], \eta \in (0,1]. \tag{16}$$

By using Theorem 5, Theorem 9, and Eq. (16), we obtain

$$\varphi(s)\, L_{q,\varphi}^{\eta}[S(t); s] - S_0$$
$$= -\lambda^{\eta}\, L_{q,\varphi}^{\eta}[S(t); s] + \lambda^{\eta} S_a\, L_{q,\varphi}^{\eta}[1; s], S(0) = S_0$$

Rearranging the above expression leads to

$$L_{q,\varphi}^{\eta}[S(t); s] = \frac{\lambda^{\eta} S_a}{\varphi(s)(\varphi(s)+\lambda^{\eta})} + \frac{S_0}{\varphi(s)+\lambda^{\eta}}. \tag{17}$$

Applying partial fraction decomposition, the transformed solution can be written as

$$L_{q,\varphi}^{\eta}[S(t); s] = \lambda^{\eta} S_a\left(\frac{1}{\lambda^{\eta}\varphi(s)} - \frac{1}{\lambda^{\eta}(\varphi(s)+\lambda^{\eta})}\right) + S_0\left(\frac{1}{\varphi(s)+\lambda^{\eta}}\right)$$
$$= S_a\left(\frac{1}{\varphi(s)}\right) + (S_0 - S_a)\left(\frac{1}{\varphi(s)+\lambda^{\eta}}\right). \tag{18}$$

Finally, by applying the inverse GF Laplace transform defined in Eq. (12) to Eq. (18), the particular solution of the GF Newton's cooling equation is obtained as

$$S(t) = S_a + (S_0 - S_a)\, exp\ exp\left(-\lambda^{\eta}\frac{(q(t))^{\eta}}{\eta}\right). \tag{19}$$

Since this solution explicitly depends on the fractional order η and the parameter q, it is denoted by

$$S_{\eta,q}(t) = S_a + (S_0 - S_a)\, exp\ exp\left(-\lambda^{\eta}\frac{(q(t))^{\eta}}{\eta}\right). \tag{20}$$

This result generalizes the classical exponential cooling behavior and reduces to the standard Newton's cooling solution when $\eta = 1$ and $q = t$.

DISCUSSION

This section examines the solution of the generalized fractional form of Newton's law of cooling, as given in Eq. (20). The behaviour of the model is illustrated using parameter values adopted from [13]. Specifically, the ambient temperature is taken as $S_a = 5°C$, the initial temperature as $S_0 = 100°C$, the observed temperature as $S_1 = 60°C$, and the observation time as $\tau = 600\ s$. Using the relation $\lambda_q(\eta) = \frac{\eta}{(q(\tau))^\eta} ln\ ln \left(\frac{S_0 - S_a}{S_1 - S_a}\right)$, we obtain

$$\lambda_q(\eta) = 0.5465 \frac{\eta}{(q(600))^\eta}. \qquad (21)$$

To investigate the influence of the fractional order and the choice of kernel function, four values of the fractional parameter are considered, namely: $\eta = 0.25, 0.5, 0.75,$ and 1. In addition, three different forms of the function $q(t)$ are employed, namely $q(t) = \sqrt{t}$, $q(t) = t$, and $q(t) = t^2$. The results corresponding to each choice of q are presented and compared.

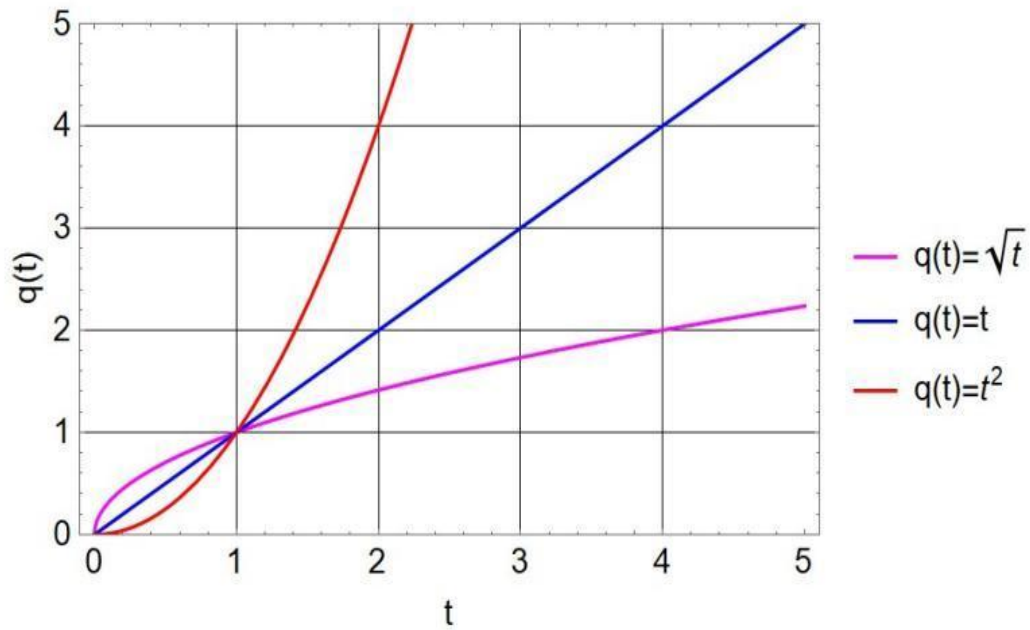

Figure 1. The graph of $q(t) = \sqrt{t}$, $q(t) = t$, and $q(t) = t^2$.

Figure 1 illustrates the growth characteristics of the three selected functions q. These functions play a crucial role in shaping the generalized time scale $\frac{(q(t))^\eta}{\eta}$, and consequently influence the thermal evolution predicted by the model. From the figure, it can be observed that over the interval $(0,1)$, the linear function $q(t) = t$ exhibits the most rapid increase, while the quadratic function $q(t) = t^2$ grows more slowly in comparison. The function $q(t) = \sqrt{t}$ lies between these two behaviours but remains closer to the horizontal axis for small values of t.

In contrast, for $t > 1$, the ordering of growth rates changes significantly. The quadratic function $q(t) = t^2$ becomes dominant and increases at a much faster rate than the other two functions, whereas $q(t) = \sqrt{t}$ grows the slowest. The linear function maintains an intermediate growth profile across this interval.

This variation in growth behaviour directly affects the cooling dynamics in the GF model. In particular, faster growth of q leads to a stronger exponential decay in the solution, resulting in a more rapid approach toward the ambient temperature. Conversely, slower growth of q yields a more gradual cooling process, highlighting the flexibility of the model in capturing different thermal response characteristics.

To further illustrate the impact of the kernel function, Eq. (20) is evaluated for three representative choices of q. We begin by considering the case $q(t) = \sqrt{t}$. Substituting Eq. (21) into Eq. (20), together with the temperature values $S_0 = 100°C$ and $S_a = 5°C$, yields

$$S_{\eta,\sqrt{t}}(t) = 5 + 95\, exp\, exp \left(-\left(0.5465\frac{\eta}{(\sqrt{600})^{\eta}}\right)\frac{(\sqrt{t})^{\eta}}{\eta}\right)$$

which is subsequently analyzed for four different fractional orders, namely $\eta = 0.25, 0.5, 0.75,$ and 1.

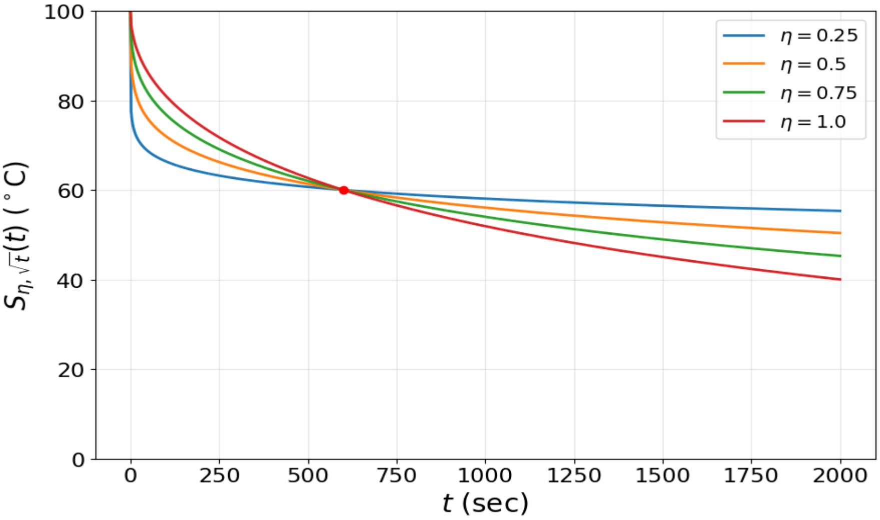

Figure 2. The graph of $S_{0.25,\sqrt{t}}(t)$, $S_{0.5,\sqrt{t}}(t)$, $S_{0.75,\sqrt{t}}(t)$, and $S_{1,\sqrt{t}}(t)$.

Figure 2 illustrates the temperature evolution corresponding to $q(t) = \sqrt{t}$. The curves show that smaller values of η lead to a noticeably slower decay toward the ambient temperature. This behavior is particularly evident at early times, where the growth of $\sqrt{t}$ remains relatively moderate, thereby delaying the effective exponential decay. As η approaches 1, the solution becomes steeper and more closely resembles the classical exponential cooling profile.

Next, we consider the case $q(t) = t$. Substituting Eq. (21) into Eq. (20), together with the parameter values $S_a = 5°C$ and $S_0 = 100°C$, yields

$$S_{\eta,t}(t) = 5 + 95\, exp\, exp \left(-\left(0.5465\frac{\eta}{(600)^{\eta}}\right)\frac{t^{\eta}}{\eta}\right).$$

This expression is then analyzed for four different fractional orders, namely $\eta = 0.25, 0.5, 0.75,$ and 1.

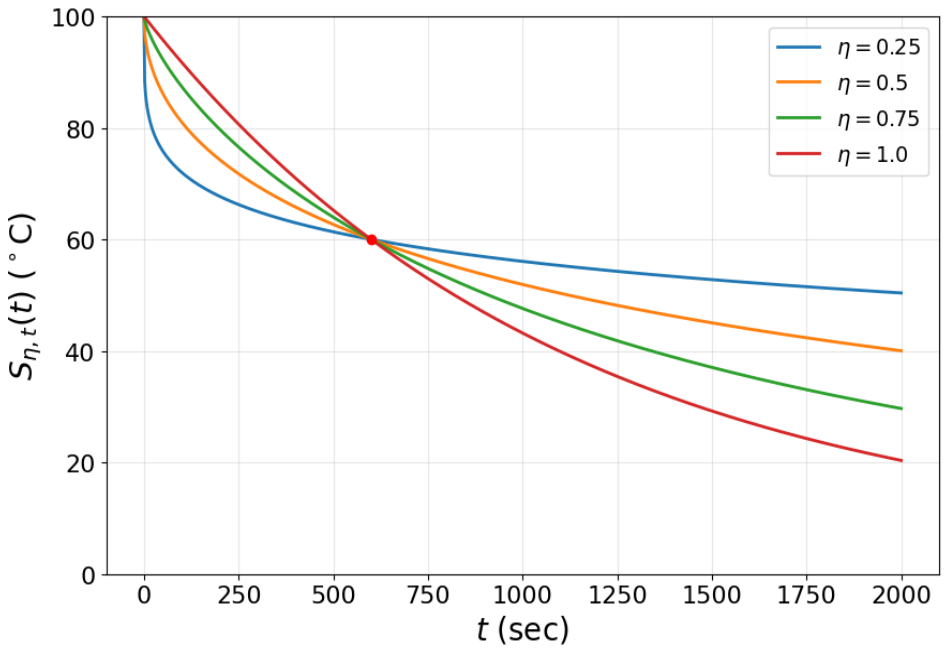

Figure 3. The graph of $S_{0.25,t}(t)$, $S_{0.5,t}(t)$, $S_{0.75,t}(t)$, and $S_{1,t}(t)$.

As shown in Figure 3, the case $q(t) = t$ produces a more balanced decay behaviour across the entire time interval. The influence of η remains significant: decreasing η slows the convergence toward equilibrium, while $\eta = 1$ recovers the classical Newtonian cooling law. Compared to the previous case, the transition toward equilibrium occurs more uniformly over time.

Finally, we consider the case $q(t) = t^2$. Substituting Eq. (21) into Eq. (20), together with the parameter values $S_0 = 100°C$ and $S_a = 5°C$, yields

$$S_{\eta,t}(t) = 5 + 95\, exp\, exp\left(-\left(0.5465\frac{\eta}{(600^2)^\eta}\right)\frac{(t^2)^\eta}{\eta}\right).$$

The resulting expression is further evaluated by considering four values of the fractional order, specifically $\eta = 0.25, 0.5, 0.75,$ and 1.

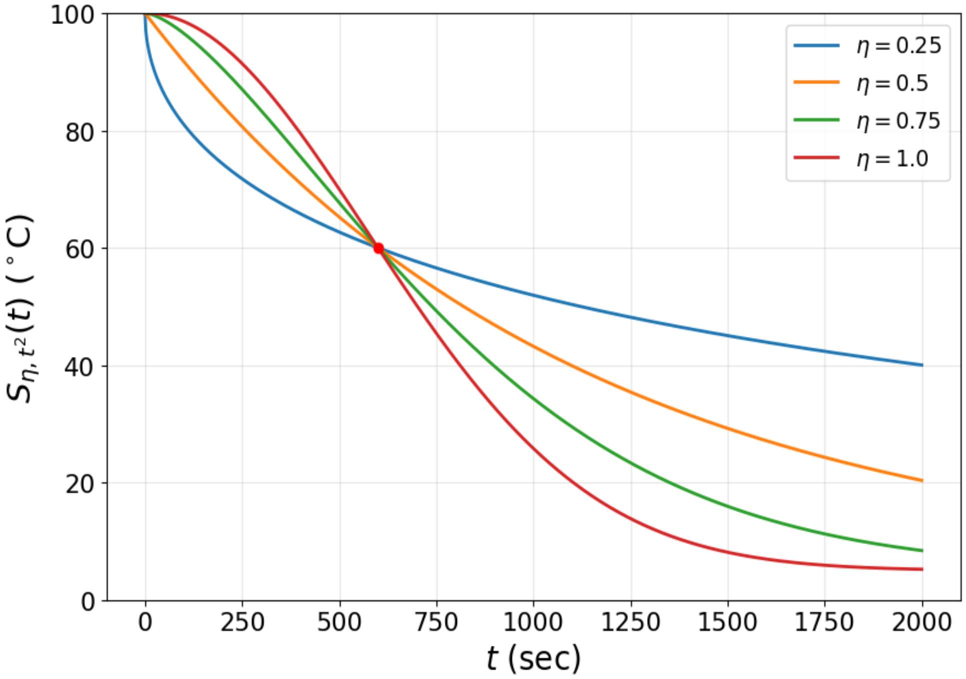

Figure 4. The graph of $S_{0.25,t^2}(t)$, $S_{0.5,t^2}(t)$, $S_{0.75,t^2}(t)$, and $S_{1,t^2}(t)$.

Figure 4 demonstrates that the quadratic kernel significantly accelerates the decay process, particularly for larger values of t. The rapid growth of t^2 amplifies the exponential term, causing the solution to approach the ambient temperature much faster than in the previous cases. Even for smaller values of η, , the cooling process becomes more pronounced at later times.

Figures 2–4 consistently indicate that decreasing the fractional order η delays the system's convergence to steady state. This reflects the increasing influence of memory effects in the fractional model, which effectively slows down the thermal response. Moreover, the choice of kernel function q plays a crucial role in shaping the cooling dynamics:

1. For $q(t) = \sqrt{t}$, the cooling process is relatively slow, especially at early times.
2. For $q(t) = t$, the behaviour aligns closely with the classical model, providing a moderate decay rate.
3. For $q(t) = t^2$, the cooling process is significantly accelerated, particularly over longer time intervals.

It is also worth noting that all solution curves intersect at a common reference point when evaluated at a fixed time (e.g., $t = 600s$), provided identical parameter values are used. This indicates that while the transient dynamics differ, the solutions can coincide under specific conditions, highlighting the sensitivity of the model to both η and q.

CONCLUSION

In this work, a generalized fractional formulation of Newton's law of cooling has been successfully established within a unified analytical framework. By introducing a novel GF Laplace transform and rigorously deriving its principal properties, the study provides a consistent and flexible tool for solving fractional differential equations involving nonlocal and memory-dependent effects.

The analytical solution obtained for the generalized fractional cooling model extends the classical exponential law by incorporating both the fractional order and a kernel function, offering a broader spectrum of dynamic behaviour. It is shown that the model recovers the classical solution as a special case, ensuring consistency with established theory while enabling richer physical interpretations.

The numerical investigation confirms that the fractional order plays a crucial role in controlling the memory effect, where lower values lead to slower thermal evolution. In addition, the kernel function significantly influences the effective time scale, with different functional forms producing distinct cooling rates. These results demonstrate that the proposed model can adapt to various thermal scenarios with higher flexibility compared to traditional approaches.

Overall, the integration of generalized fractional calculus with the Laplace transform method provides a powerful analytical framework for modelling heat transfer processes. Future work may explore its application to more complex systems and experimental validation to further assess its practical relevance.

ACKNOWLEDGMENTS

The authors would like to express their sincere gratitude to Sebelas Maret University for supporting this research during the 2024 academic year through the research grant under Contract No. 194.2/UN27.22/PT.01.03/2024.

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